1. Consider the sequence

\[ c_n = \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}. \]

(a) Show that \((c_n)\) is monotonic.

(b) Show that \((c_n)\) is bounded.

(c) Based on (a) and (b), what can we conclude about \((c_n)\)?

2. Let \(a, b \in (0, \infty)\) be given. Define the sequence \((x_n)\) recursively by

\[ x_1 = a, \quad x_{n+1} = \frac{x_n}{2} + \frac{b}{2x_n}. \]

Define also \(y_n = \frac{x_n - \sqrt{b}}{x_n + \sqrt{b}}\).

(a) Use the recursion for \(x_n\) to prove the new recursion \(y_{n+1} = y_n^2\).

(b) Use (a) to find a closed expression for \(y_n\) in terms of \(a, b\) and \(n\).

(c) Use (b) to compute \(\lim_{n \to \infty} x_n\). Your answer should depend only on the parameters \(a, b\).

(Hint: (b) Find \(y_2\) in terms of \(y_1\). How about \(y_3\)? Do you see a pattern?
(c) Solve for \(x_n\) in terms of \(y_n\). Use (b) to compute the limit of \(y_n\) first and then apply appropriate limit laws.)

3. A right triangle \(ABC\) has the angle \(\hat{C} = 90^\circ\) and \(\hat{A} = \theta \in (0, 90^\circ)\). The length of the side \(AC = b\). Consider \(C_1\) to be the point on \(AB\) such that \(CC_1\) is perpendicular to \(AB\), \(C_2\) the point on \(BC\) such that \(C_1C_2\) is perpendicular to \(BC\), \(C_3\) the point on \(AB\) such that \(C_2C_3\) is perpendicular to \(AB\), and so on (meaning, this process continues indefinitely). Show that the total length of all the perpendiculars

\[ CC_1 + C_1C_2 + C_2C_3 + \ldots \]

is finite and find its exact value in terms of \(b\) and \(\theta\).

(Hint: Start by drawing a picture. In the right triangle \(ABC\), how do you compute \(CC_1\)?)

4. Compute the exact value of the sum

\[ \sum_{n=1}^{\infty} \frac{n}{(1 + \sqrt{n+1})\sqrt{(n+1)!}}. \]
(Hint: Manipulating the general term (algebra!), this turns out to be a telescoping series.)

5. Consider the sequence

\[ x_n = \sum_{k=1}^{n} \frac{\sin(k)}{k + n^2} = \frac{\sin(1)}{1 + n^2} + \frac{\sin(2)}{2 + n^2} + \cdots + \frac{\sin(n)}{n + n^2}. \]

Use the \( \epsilon \)-definition of limit to show that \( \lim_{n \to \infty} x_n = 0. \)

(Hint: Find a better inequality for \( |x_n| \).)