Limits and Infinite Series
Lecture Notes for Math 226

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To the student

Math 226 is a first introduction to formal arguments in mathematical analysis that is centered around the concept of limit. You have already encountered this concept in your calculus classes, but now you will see it treated from an abstract (and more rigorous) point of view.

A main goal of this course if to help you develop an ability of clearly, correctly and completely presenting your mathematical reasoning. You will learn to construct your own proofs by thinking about the definitions, the proofs of theorems or examples, and solving additional exercises from the notes. Do not treat this manuscript as a novel. To be successful, an “engaged” reading with pencil and paper is crucial.

The lecture notes were written and were first used as a substitute textbook for the classes I taught in the Spring and Summer of 2007 at Western Washington University. The current pages are an improved version of the initial set of notes in which I corrected mathematical errors or typos, added several examples and exercises, and included a couple more sections that relate well to the “core” material.

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1.1. Infinite limits at infinity

Infinity is a concept that we find convenient to use when describing the behavior of a function whose input or output values become arbitrarily large. We will distinguish between positive infinity (+∞) and negative infinity (−∞): +∞ or simply ∞ has to do with a quantity that becomes arbitrarily large and positive, −∞ with some quantity that becomes arbitrarily large and negative. It is essentially the same as comparing the numbers 2^{2007} and −2^{2007}, which are both large but of opposite signs.

Suppose that, for some real number \( a \), the real-valued function \( f(x) \) is defined for all values \( x > a \); in this case, we say that \( f \) is defined on the open interval \((a, \infty)\), and write \( f : (a, \infty) \rightarrow \mathbb{R} \).

The informal definition of \( \lim_{x \to \infty} f(x) = \infty \) should tell us then that as \( x \) increases without bound, \( f(x) \) itself becomes arbitrarily large. That is, we can make \( f(x) \) as large as we like by choosing \( x \) sufficiently large. While we might have an intuition of what “sufficiently large” or “arbitrarily large” means, we need to turn the informal statement above into one that tells us exactly what we ought to check. This leads us to the precise definition.

**Definition 1.1.1.** Let \( a \) be a real number and \( f : (a, \infty) \rightarrow \mathbb{R} \) be a function. We say that the limit of \( f(x) \) as \( x \) approaches \( \infty \) is \( \infty \), and write

\[
\lim_{x \to \infty} f(x) = \infty,
\]

if, for every real number \( B \), there exists a real number \( A \geq a \) such that for all \( x > A \), it is true that \( f(x) > B \).

In analogy with Definition 1.1.1 above, we can also precisely define the limit concepts that have to do with \( -\infty \). Note that in each definition we need to change the domain of the function \( f \) accordingly, so that we can talk about arbitrarily large positive or negative values of \( x \).

**Definition 1.1.2.** Let \( a \) be a real number and \( f : (a, \infty) \rightarrow \mathbb{R} \) be a function. We say that the limit of \( f(x) \) as \( x \) approaches \( \infty \) is \( -\infty \), and write

\[
\lim_{x \to \infty} f(x) = -\infty,
\]
if, for every real number $B$, there exists a real number $A \geq a$ such that for all $x > A$, it is true that $f(x) < B$.

**Definition 1.1.3.** Let $b$ be a real number and $f : (-\infty, b) \to \mathbb{R}$ be a function. We say that the limit of $f(x)$ as $x$ approaches $-\infty$ is $\infty$, and write

\[
\lim_{x \to -\infty} f(x) = \infty,
\]

if, for every real number $B$, there exists a real number $A$ such that for all $x < A$, it is true that $f(x) > B$.

**Definition 1.1.4.** Let $a$ be a real number and $f : (-\infty, a) \to \mathbb{R}$ be a function. We say that the limit of $f(x)$ as $x$ approaches $-\infty$ equals $-\infty$, and write

\[
\lim_{x \to -\infty} f(x) = -\infty,
\]

if, for every real number $B$, there exists a real number $A \leq a$ such that for all $x < A$, it is true that $f(x) < B$.

When trying to use the limit definitions above in a specific example, the following “guiding points” help in properly organizing your argument or proof.

1. Start your proof by fixing $B$. This can be expressed in words as: *Let $B$ (real) be given.*
2. The number $A$ in the definition will depend on the given $B$. Your argument will need to show specifically this dependence.
3. In principle, the choice of $A$ should be such that the values $x > A$ fall within the domain of the function $f$. For simplicity, in all the examples below the domain of the function is the set of all real numbers $\mathbb{R}$, thus this assumption can be ignored.

It is extremely important that you work through these definitions on simple examples (such as the ones listed below) until you come to understand their meaning. Simply memorizing them without actually seeing for yourself how to explicitly find the value $A$ in terms of the given $B$ will not shed much light on the deep concept of limit. There are more complicated examples in which it is rather difficult to explicitly find $A$ in terms of $B$; these examples are usually dealt with via general theorems about limits.

**Example 1.1.5.** Show that \( \lim_{x \to -\infty} (x - 2007) = \infty. \)

**Proof.** We will prove that all the requirements in Definition 1.1.1 are satisfied. In this case the function considered is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - 2007$; see (3).

Using the “guiding point” (1) above, we let $B$ be a given real number. Then, following the “guiding point” (2) above, we choose $A = B + 2007$. Now, it is easy to see that:

if $x > A$, then $f(x) = x - 2007 > A - 2007 = B$,

and we are done. \( \square \)
But how did we arrive at our choice of $A$ in terms of the “fixed $B$”? Perhaps it is instructive to ask ourselves what happens if we start with the inequality $f(x) > B$. Thus, for a given $B \in \mathbb{R}$, we want

$$f(x) > B \iff x - 2007 > B \iff x > B + 2007.$$ 

Remember that we are looking for an $A$ such that $x > A \Rightarrow f(x) > B$. Clearly, the choice $A = B + 2007$ will do.

**Question 1.1.6.** Is the choice of $A = B + 2007$ the only one for which the implication $x > A \Rightarrow f(x) > B$ holds? What is particular about the choice $A = B + 2007$?

**Example 1.1.7.** Show that $\lim_{x \to \infty} (3x + e^{-x}) = \infty$.

**Proof.** Let $B$ be given. Choose $A = B/3$ and suppose that $x > A$. Then

$$f(x) = 3x + e^{-x} > 3x > 3A = B.$$

Note that we used the fact that $e^{-x} > 0$ for all real values of $x$ and this has greatly simplified our work. Indeed, if instead we would have taken the second approach in the previous example and would have started with $f(x) > B$, we would have faced the problem of solving the inequality $3x + e^{-x} > B$, and this is hard to do! However, since $3x + e^{-x} > 3x$, it is sufficient to solve $3x > B$, which gives $x > B/3 = A$.

**Question 1.1.8.** What is a possible direct approach to solve $3x + e^{-x} > B$?

Before we state the next example, it is useful to introduce the following definition.

**Definition 1.1.9.** Let $a, b$ be two real numbers. Their maximum, denoted by $\max(a, b)$, is given by $\max(a, b) = a$ if $a \geq b$, and $\max(a, b) = b$ if $a \leq b$. The minimum, $\min(a, b)$, is given by $\min(a, b) = -\max(-a, -b)$.

**Question 1.1.10.** Based on Definition 1.1.9, can you show that $\min(a, b) = b$ if $a > b$?

**Example 1.1.11.** Show that $\lim_{x \to \infty} (x^2 - x) = \infty$.

**Proof.** Let $B$ be given. Choose $A = \max(B, 2)$ and suppose that $x > A$. Note that, by our choice, $x > 0$ and, in particular, $x > B$ and $x - 1 > 1$. Hence, we can write

$$x^2 - x = x(x - 1) > x \cdot 1 > B.$$
It is instructive to try to find \( A \) directly from the inequality \( f(x) > B \) in Example 1.1.11. This is equivalent to \( x^2 - x - B > 0 \), which is a quadratic inequality that we know how to solve. Indeed, if we use the quadratic formula we arrive at
\[
x > \frac{1 + \sqrt{1 + 4B}}{2}.
\]
So, if we pick \( A = \frac{1 + \sqrt{1 + 4B}}{2} \), we should be done. But choosing this \( A \) requires the existence of the square root and for that we need \( B \geq -1/4 \)!

What happens if \( B < -1/4? \) As it turns out this case is much easier, since we can write
\[
x^2 - x - B > x^2 - x + 1/4 = (x - 1/2)^2 \geq 0,
\]
for all real \( x \) (that is we can pick any \( A \) for this case).

The moral of the previous example is that in Definition 1.1.1, it is sufficient to let \( B \) be a given positive real number, and to find a positive real number \( A \) for which the definition is satisfied. This makes sense, since we are really interested in the behavior of large positive \( x \) and large positive \( f(x) \).

**Question 1.1.12.** What assumptions could you make in Definitions 1.1.2-1.1.4 about the signs of \( A, B \) that would not affect their statements?

You might also ask yourself: which route do I need to take in finding the dependence of \( A \) on \( B? \) To answer this question, you must first realize that there is a significant difference between trying to solve \( f(x) > B \) in Examples 1.1.5 or 1.1.11 and solving the inequality \( f(x) > B \) in Example 1.1.7.

**Exercises**

In all exercises below you will need to use the precise definition of limit. Graphing the functions can give you an intuition of why the statement should be true, but you are expected to provide rigorous proofs along the lines outlined in the Examples 1.1.5-1.1.11.

**Exercise 1.1.13.** Prove that \( \lim_{x \to \infty} 2007x = \infty \).

**Exercise 1.1.14.** Let \( B = 4014 \).
(a) Find an \( A \) such that for all \( x > A \) it is true that \( 2007x > B \).
(b) Find another \( \tilde{A} \neq A \) such that for all \( x > \tilde{A} \) it is true that \( 2007x > B \).
(c) What is the smallest choice of \( A \) for which \( x > A \iff 2007x > B? \)

**Exercise 1.1.15.** Prove that \( \lim_{x \to \infty} x^2 = \infty \).

**Exercise 1.1.16.** Prove that \( \lim_{x \to \infty} x^3 = \infty \).

**Exercise 1.1.17.** Prove that \( \lim_{x \to \infty} (x^3 - x^2) = \infty \).

**Exercise 1.1.18.** Prove that \( \lim_{x \to \infty} (\sqrt{x} - \sqrt[3]{x}) = \infty \).

**Exercise 1.1.19.** Prove that \( \lim_{x \to \infty} (\frac{x}{2} + \cos x) = \infty \).


Exercise 1.1.20. Prove that \( \lim_{x \to \infty} (2x - e^{-x}) = \infty \).

Exercise 1.1.21. Prove that \( \lim_{x \to \infty} \frac{x^2}{x - 1} = \infty \).

Exercise 1.1.22. Prove that \( \lim_{x \to \infty} \frac{x^2(2 + \sin x)}{x + \cos x} = \infty \).

Exercise 1.1.23. Prove that \( \lim_{x \to -\infty} x^2 = \infty \).

Exercise 1.1.24. Prove that \( \lim_{x \to -\infty} x^3 = -\infty \).

Exercise 1.1.25. Prove that \( \lim_{x \to -\infty} \frac{x^2}{x - 1} = -\infty \).

1.2. Finite limits at infinity

We are trying to establish the rigorous definition of horizontal asymptote of a given function \( f \). Intuitively, we expect a horizontal asymptote to occur if the distance between the graph of \( f \) and some fixed line becomes increasingly smaller as we move along the graph increasingly far from the origin. This is the case if we consider the function \( f(x) = \frac{1}{x} \) (for \( x \neq 0 \)). If \( x > 0 \) becomes increasingly large, then \( 1/x \) becomes increasingly small (and positive), while when \( x < 0 \) and its magnitude becomes increasingly large, \( 1/x \) becomes again increasingly small (and negative). Therefore, a simple inspection of the behavior of \( 1/x \) for values of \( x \) that are large tells us that the line \( y = 0 \) (the \( x \)-axis) is a horizontal asymptote. We can also summarize this by saying that the limit of \( f(x) = 1/x \) as \( x \) approaches \( \infty \) or \( -\infty \) is 0, which we would write

\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0.
\]

In general, if \( L \) is some real number, the informal definition of

\[
\lim_{x \to \infty} f(x) = L
\]

should tell us that as \( x \) gets (positively) large, \( f(x) \) approaches \( L \), that is, by choosing \( x \) sufficiently large, \( f(x) \) gets arbitrarily close to \( L \). The definition of the limit at \( -\infty \) would be similar, except that now \( x \) becomes large in magnitude, but it is negative. Now the question is: what do we mean by two real numbers to be “arbitrarily close”? To answer it we need the concept of distance between two numbers.

Definition 1.2.1. Let \( a, b \) be two real numbers. The distance between \( a \) and \( b \) is given by the absolute value of their difference, \( |a - b| \).

You should have encountered the definition of the absolute value of a real number before; some of its important properties are summarized in Exercise 1.2.8. In particular, we have that the distance between \( a \) and \( b \) is the same as the distance between \( b \) and \( a \).
Going back to our question, we would then say that \( a \) is “arbitrarily close” to \( b \) if their distance \( |a - b| \) is “arbitrarily close” to 0. But, we still have not defined “arbitrarily close”! What we would like this to mean is that the distance between the two numbers \( a, b \) can be made less than any prescribed (positive) error, no matter how small. While this might be intuitively clear, we would like a precise definition.

**Definition 1.2.2.** Let \( a \) be a real number and \( f : (a, \infty) \rightarrow \mathbb{R} \) be a function. We say that the limit of \( f(x) \) as \( x \) approaches \( \infty \) equals the real number \( L \), and write
\[
\lim_{x \to \infty} f(x) = L,
\]
if, for every \( \epsilon > 0 \), there exists a real number \( A \geq a \) such that for all \( x > A \), it is true that \( |f(x) - L| < \epsilon \).

Similarly, we have

**Definition 1.2.3.** Let \( a \) be a real number and \( f : (-\infty, a) \rightarrow \mathbb{R} \) be a function. We say that the limit of \( f(x) \) as \( x \) approaches \(-\infty \) equals the real number \( L \), and write
\[
\lim_{x \to -\infty} f(x) = L,
\]
if, for every \( \epsilon > 0 \), there exists a real number \( A \leq a \) such that for all \( x < A \), it is true that \( |f(x) - L| < \epsilon \).

As we have already noticed in the previous section, in the Definition 1.2.2 we can choose \( A > 0 \) (since \( x \) becomes increasingly large and positive), while in Definition 1.2.3 we really care about \( A < 0 \).

In a specific argument argument, we will start with a given \( \epsilon \) - this is the prescribed error we alluded to before- and write: *Let \( \epsilon > 0 \) be given.* The core of the proof will be to find an \( A \geq a \) which, importantly, depends on \( \epsilon \) and for which the corresponding definition holds.

**Example 1.2.4.** Show that \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

**Proof.** In this case \( f(x) = 1/x, f : (0, \infty) \rightarrow \mathbb{R} \) (thus \( a = 0 \)) and \( L = 0 \). Let \( \epsilon > 0 \) be given. Choose \( A = 1/\epsilon > 0 \). Then, for \( x > A > 0 \), we have
\[
|f(x) - 0| = \left| \frac{1}{x} \right| = \frac{1}{x} < \frac{1}{\epsilon} = \epsilon,
\]
and we are done. Note that as the prescribed error \( \epsilon \) becomes small, \( A \) becomes large.

But how could we find out what \( A \) should be? We start with the inequality we want to achieve and try to solve it! Namely,
\[
|f(x) - L| < \epsilon \Leftrightarrow \left| \frac{1}{x} - 0 \right| < \epsilon \Leftrightarrow |x| > \frac{1}{\epsilon}.
\]
Since we only care about \( x > 0 \) in this case, the last inequality gives \( x > 1/\epsilon = A \). Note that, in fact, any positive number larger than \( 1/\epsilon \) is a valid choice of \( A \). We have found the smallest one that works.

**Example 1.2.5.** Show that \( \lim_{x \to -\infty} \frac{1}{x} = 0 \).

**Proof.** We proceed as before. Now, \( f(x) = 1/x \) but \( f : (-\infty, 0) \to \mathbb{R} \) (thus, still, \( a = 0 \)). Let \( \epsilon > 0 \) be given. Choose \( A = -1/\epsilon < 0 \). Let \( x < A < 0 \), we have

\[
|f(x)| = \left| \frac{1}{x} \right| = - \frac{1}{x} < - \frac{1}{1/\epsilon} = \epsilon,
\]

and we are done. \( \square \)

**Question 1.2.6.** Why does the inequality \( x < -1/\epsilon \) imply \( -1/x < -1/\epsilon^2 \)? Make sure you understand all the algebraic steps involved. Also, can you find \( A \) by directly solving the inequality \( |f(x) - L| < \epsilon \)?

**Example 1.2.7.** Show that \( \lim_{x \to \infty} \frac{x}{x + 1} = 1 \).

**Proof.** In this case \( f(x) = x/(x + 1) \), say \( f : (-1, \infty) \to \mathbb{R} \), and \( L = 1 \). Let \( \epsilon > 0 \) be given. Choose \( A = 1 - 1 > -1 \) and suppose that \( x > A \). Then

\[
\left| \frac{x}{x + 1} - 1 \right| = \left| \frac{-1}{x + 1} \right| = \frac{1}{x + 1} < \frac{1}{A + 1} = \frac{1}{1/\epsilon} = \epsilon.
\]

\( \square \)

How did we find \( A \) in terms of \( \epsilon \)? We first noted that we only care about values of \( x > -1 \) (we are finding a limit at positive infinity). Therefore, if we try to solve the inequality \( |f(x) - L| < \epsilon \), we arrive as before at

\[
\frac{1}{x + 1} < \epsilon \Leftrightarrow x > \frac{1 - \epsilon}{\epsilon} = A.
\]

**Exercises**

In all exercises below you will need to use the precise definition of limit. Graphing the functions can give you an intuition of why the statement should be true, but you are expected to provide rigorous proofs along the lines outlined in the Examples 1.2.4-1.2.7.

**Exercise 1.2.8.** Given a real number \( x \), its absolute value is defined by \( |x| = x \) if \( x \geq 0 \), and \( |x| = -x \) if \( x < 0 \). Let \( a, b, c \) be three arbitrary real numbers.

(a) Show that \( |a| = |-a| \).
(b) Prove the triangle inequality \( |a + b| \leq |a| + |b| \).
(c) Use (b) to prove the reverse triangle inequality \( ||a| - |b|| \leq |a - b| \).
(d) Use (b) to prove that \( |a - b| \leq |a - c| + |c - b| \).
Exercise 1.2.9. Let \( f : (0, \infty) \to \mathbb{R} \), \( f(x) = 2007 + \frac{1}{x} \) and \( \epsilon = 0.5 \).

(a) Find \( L \) and an \( A \) such that for all \( x > A \) it is true that \( |f(x) - L| < \epsilon \).
(b) Find another \( \tilde{A} \neq A \) such that for all \( x > \tilde{A} \) we have \( |f(x) - L| < \epsilon \).
(c) What is the smallest choice of \( A \) for which \( x > A \iff |f(x) - L| < \epsilon \)?

Exercise 1.2.10. Prove that \( \lim_{x \to -\infty} \left( 2007 + \frac{1}{x} \right) = 2007 \).

Exercise 1.2.11. Prove that \( \lim_{x \to \infty} \frac{\pi}{x^2 + 7} = 0 \).

Exercise 1.2.12. Prove that \( \lim_{x \to \infty} \frac{\cos x}{x} = 0 \).

Exercise 1.2.13. Prove that \( \lim_{x \to -\infty} \frac{\cos x + x}{x} = 1 \).

Exercise 1.2.14. Prove that \( \lim_{x \to \infty} \frac{2x}{x + 1} = 2 \).

Exercise 1.2.15. Prove that \( \lim_{x \to \infty} \frac{2x + \cos x}{x + 1} = 2 \).

Exercise 1.2.16. Prove that \( \lim_{x \to -\infty} \frac{3x}{x + 2} = 3 \).

Exercise 1.2.17. Prove that \( \lim_{x \to \infty} (\sqrt{x^2 + 1} - x) = 0 \).

1.3. Universal and existential quantifiers. Definitions revisited

A predicate \( p \) is a statement that may be true or false depending on the values of its variable \( x \). \( p(x) \) is also referred to as a propositional function because each choice of \( x \) produces a proposition that is true or false.

The universal quantifier \( \forall \) has the meaning for all or for every. The sentence for all \( x, p(x) \) can be simply written as \( \forall x, p(x) \).

Example 1.3.1. The variable \( x \) has a different meaning in different circumstances, therefore we specify it.

(1) \( \forall \) people \( x \) having zip code 98225, \( x \) is a resident of Bellingham.
(2) \( \forall \) human beings \( x \) born in 1972, age of \( x \) is greater than 38 years.
(3) \( \forall x \) real, \( x^2 \geq 0 \).

The existential quantifier \( \exists \) has the meaning there exists or there is. The sentence there exists \( x \) such that \( p(x) \) can be simply written as \( \exists x, p(x) \).

Example 1.3.2. As in Example 1.3.1, we specify the meaning of the variable.

(1) \( \exists \) a person \( x \), \( x \) is a Math 226 student.
(2) \( \exists \) a person \( x \) born in 1973, age of \( x \) is less than 37.
(3) \( \exists x \) real, \( x^2 = 4 \).
To negate a sentence, we use the quantifier \( \neg \) in front of it. So, \( \neg p(x) \) is simply written \( \neg p(x) \). When we negate a sentence, any of the two quantifiers above is transformed into the other one, that is:

1. \( \neg (\forall x, p(x)) \) means \( \exists x, \neg p(x) \)
2. \( \neg (\exists x, p(x)) \) means \( \forall x, \neg p(x) \).

For example, if we negate Example 1.3.1 (c), we get

\[ \neg (\forall x \text{ real}, x^2 \geq 0) \equiv \exists x \text{ real}, x^2 < 0 \]

while if we negate Example 1.3.2 (c), we get

\[ \neg (\exists x \text{ real}, x^2 = 4) \equiv \forall x \text{ real}, x^2 \neq 4 \]

We can now restate all our previous definitions in terms of the learned quantifiers. For example, Definition 1.2.2 can be simply stated as

\[ \lim_{x \to \infty} f(x) = L \iff \forall \epsilon > 0, \exists A \geq a \text{ (such that)} \forall x > A, |f(x) - L| < \epsilon. \]

Also, saying that \( \lim_{x \to \infty} f(x) = L \) is false or not true, means that the negation of the above statement

\[ \neg (\lim_{x \to \infty} f(x) = L) \]

is true. This is equivalent to

\[ \exists \epsilon > 0 \text{ (such that)} \forall A \geq a, \exists x > A \text{ (such that)} |f(x) - L| \geq \epsilon \]

is true.

**Question 1.3.3.** How can we rephrase the statement “\( f(x) \) does not have a finite limit at \( \infty \)” using quantifiers?

**Exercises**

In all exercises below you are implicitly asked to make use of the symbols \( \forall, \exists \) and \( \neg \) discussed above.

**Exercise 1.3.4.** Negate the statements of Definitions 1.1.1-1.1.4.

**Exercise 1.3.5.** Show that \( \lim_{x \to \infty} \cos x = 0 \) is false.

**Exercise 1.3.6.** Show that \( \lim_{x \to \infty} \cos x = 0.5 \) is false.

**Exercise 1.3.7.** Show that \( \lim_{x \to \infty} \cos x = -2 \) is false.

**Exercise 1.3.8.** Show that “\( \lim_{x \to \infty} \cos x \) exists and is finite” is a false statement.

**Exercise 1.3.9.** Find two functions \( f \) and \( g \) such that \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to \infty} g(x) \) do not exist, but \( \lim_{x \to \infty} (f(x) + g(x)) \) exists and is finite.

**Exercise 1.3.10.** Is it possible to have two functions \( f \) and \( g \) such that \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to \infty} g(x) \) do not exist, but \( \lim_{x \to \infty} (f(x)g(x)) \) exists and is finite?
Exercise 1.3.11. Prove that if \( \lim_{x \to \infty} f(x) = \infty \), then the statement “\( \lim_{x \to \infty} f(x) \) exists and is finite” is false.

Exercise 1.3.12. Investigate the existence of the limit \( \lim_{x \to -\infty} \frac{1}{\left| \sin \left( \frac{1}{x} \right) \right|} \).
CHAPTER 2

Limits at a point

2.1. Finite limit at a point

Suppose that the function \( f(x) \) is defined for all inputs \( x \) in an open interval about a real number \( x_0 \) (fixed, for now), and possibly only for \( x \neq x_0 \). That is to say, we care about the values of \( x \) around \( x_0 \), but not particularly about the value \( f(x_0) \). Suppose also that \( L \) is some real number.

The informal definition of

\[
\lim_{x \to x_0} f(x) = L
\]

should tell us that as \( x \) approaches \( x_0 \), \( f(x) \) approaches \( L \). That is, we can make \( f(x) \) arbitrarily close to \( L \) by choosing \( x \) sufficiently close to \( x_0 \) and \( x \neq x_0 \). Having experienced already with limits at infinity, you are well aware that some of the phrases we used, such as “arbitrarily close”, depend on the context. This calls for the precise \( \epsilon - \delta \) definition.

**Definition 2.1.1.** Let \( x_0 \in \mathbb{R} \) and suppose that for some real number \( \delta_0 > 0 \), \( f \) is defined on \((x_0 - \delta_0, x_0) \cup (x_0, x_0 + \delta_0)\). Let also \( L \in \mathbb{R} \). We say that the limit of \( f(x) \) as \( x \) approaches \( x_0 \) equals \( L \), and write

\[
\lim_{x \to x_0} f(x) = L,
\]

if, for every \( \epsilon > 0 \), there exists \( \delta \in (0, \delta_0] \) such that for all \( x \) with \( 0 < |x - x_0| < \delta \), it is true that \( |f(x) - L| < \epsilon \).

In terms of the quantifiers learned in the last section of the previous chapter, we can re-write the definition of \( \lim_{x \to x_0} f(x) = L \) as follows:

\[
\forall \epsilon > 0, \exists \delta \in (0, \delta_0] \text{ (such that) } \forall x, \text{ (such that) } 0 < |x-x_0| < \delta, \text{ it is true that } |f(x) - L| < \epsilon.
\]

Without absolute values, the definition looks like this:

\[
\forall \epsilon > 0, \exists \delta \in (0, \delta_0] \forall x : x_0 - \delta < x < x_0 + \delta, x \neq x_0, L - \epsilon < f(x) < L + \epsilon.
\]

Similarly to what you have already seen in Chapter 1, a proof of a limit will start with a given \( \epsilon > 0 \) and then try to identify a positive number \( \delta \) that depends on \( \epsilon \) such that the definition holds.

**Example 2.1.2.** Show that \( \lim_{x \to 1} (2x - 3) = -1 \).
Proof. In this case \( f(x) = 2x - 3, \ x_0 = 1, \ L = -1. \) Since \( f \) is defined on \( \mathbb{R}, \) we can take any positive number for \( \delta_0. \) Let \( \epsilon > 0 \) be given. Choose \( \delta = \epsilon/2 > 0 \) and suppose that \( 0 < |x - 1| < \delta. \) Then
\[
|f(x) - L| = |(2x - 3) - (-1)| = |2(x - 1)| < 2\delta = \epsilon.
\]

How do we know what \( \delta \) to choose? Start with the inequality
\[
|f(x) - L| < \epsilon \iff 2|x - 1| < \epsilon \iff |x - 1| < \frac{\epsilon}{2} = \delta.
\]
Of course any \( \tilde{\delta} < \epsilon/2 \) will also work.

Question 2.1.3. In Example 2.1.2, the function \( f \) is defined at \( x_0 = 1 \) and \( f(x_0) = -1 = L. \) It is tempting to infer that
\[
\lim_{x \to x_0} f(x) = L \iff f(x_0) = L.
\]
Why is this statement wrong? Can you modify the example above so that this equivalence is false?

Example 2.1.4. Show that \( \lim_{x \to 2} x^2 = 4. \)

Proof. Here, \( f(x) = x^2, \ x_0 = 2, \ L = 4. \) Let \( \epsilon > 0 \) be given. Let us see if the direct approach works, that is if we can figure out the \( \delta \) starting with the inequality \( |f(x) - L| < \epsilon. \) The inequality is equivalent to
\[
|x^2 - 4| < \epsilon \iff |x - 2||x + 2| < \epsilon.
\]
We should now isolate \( |x - 2| \) and to do so we need to do something about \( |x + 2| \) (hence those question marks above!). Since we only care about values of \( x \) close to 2 we can assume that \( x \) lies in the interval \((1, 3).\) We eventually want \( 0 < |x - 2| < \delta, \) so, if necessary, we can choose (later) \( \delta < \delta_0 := 1 \) which forces \( x \) to be in \((1, 3).\) But \( x \) being in this interval implies \( |x + 2| < 5. \) This is exactly what allows us to get rid of the question marks:
\[
|x - 2||x + 2| < 5|x - 2| < \epsilon \iff |x - 2| < \frac{\epsilon}{5}.
\]
This gives the choice of \( \delta = \min(1, \epsilon/5). \) The minimum is required since we need both \( |x - 2| < \epsilon/5 \) and \( |x - 2| < 1 \) (which is needed in order to write \( |x + 2| < 5 \)) to guarantee \( |f(x) - 4| < \epsilon. \) 

Example 2.1.5. Show that \( \lim_{x \to 3} \frac{1}{x} = \frac{1}{3}. \)

Proof. Let \( \epsilon > 0 \) be given. It will be useful to consider \( x \) around the value of \( x_0 = 3, \) so let us fix \( \delta_0 = 2; \) in particular, we care about \( f \) being defined on \((1, 5).\) Choose \( \delta = \min(2, 3\epsilon) \leq \delta_0 \) and suppose \( 0 < |x - 3| < \delta. \) Since \( x > 1 \) and \( |x - 3| < 3\epsilon, \) we have
\[
|\frac{1}{x} - \frac{1}{3}| = \frac{|x - 3|}{3x} < \frac{|x - 3|}{3} < \frac{3\epsilon}{3} = \epsilon.
\]
**Question 2.1.6.** In Examples 2.1.4 and 2.1.5, how would you find the δ directly? More precisely, how would you solve for x the inequalities \( L - \epsilon < f(x) < L + \epsilon \)?

We alluded to this already in Question 2.1.3: in the definition of the limit of a function \( f \) at a point \( x_0 \), the value \( f(x_0) \) does not need to exist. However, if this value exists and the limit \( L \) of a function \( f \) at \( x_0 \) exists and it equals \( f(x_0) \), then we will say that \( f \) is a continuous function at \( x_0 \). Here is the precise definition of continuity at a point.

**Definition 2.1.7.** Let \( x_0 \in \mathbb{R} \) and suppose that for some real number \( \delta_0 > 0 \), \( f \) is defined on \((x_0 - \delta_0, x_0 + \delta_0)\). We say that \( f \) is continuous at \( x_0 \) if

\[
\lim_{x \to x_0} f(x) = f(x_0),
\]

that is, if, for every \( \epsilon > 0 \), there exists \( \delta \in (0, \delta_0] \) such that for all \( x \) with \( |x - x_0| < \delta \), it is true that \( |f(x) - f(x_0)| < \epsilon \).

Moreover, if the function \( f \) is defined on some domain \( I \) and it is continuous at all points \( x_0 \) in \( I \), we say that \( f \) is continuous on \( I \).

**Exercises**

In all exercises below you will need to use the precise definition of limit. Graphing the functions can give you an intuition of why the statement should be true, but you are expected to provide rigorous proofs along the lines outlined in the Examples 2.1.2-2.1.5.

**Exercise 2.1.8.** Use quantifiers to negate Definition 2.1.1.

**Exercise 2.1.9.** Let \( f(x) = 4x + 7, x \neq 0 \) and \( f(0) = \pi \).

(a) Graph the function \( f \).
(b) Explain in your own words why \( \lim_{x \to 0} f(x) \neq \pi \).
(c) Show that \( \lim_{x \to 0} f(x) = 7 \).
(d) Is \( f \) continuous on the set of all real numbers \( \mathbb{R} \)?

**Exercise 2.1.10.** Let \( \epsilon = 0.314 \).

(a) Find a \( \delta \) such that for all \( |x| < \delta \) it is true that \( |(4x + 7) - 7| < \epsilon \).
(b) Find another \( \tilde{\delta} \neq \delta \) such that for all \( |x| < \tilde{\delta} \) it is true that \( |(4x + 7) - 7| < \epsilon \).
(c) What is the largest choice of \( \delta \) for which \( |x| < \delta \Leftrightarrow |(4x + 7) - 7| < \epsilon \)?

**Exercise 2.1.11.** Prove that \( \lim_{x \to -2} x^3 = -8 \).

**Exercise 2.1.12.** Find an \( L \) such that \( \lim_{x \to -2} (x^2 - 2x + 4) = L \). Prove this equality using the \( \epsilon - \delta \) definition for the choice of \( L \) that you make.
Exercises

Exercise 2.1.13. Find an $L$ such that \( \lim_{x \to -2} \frac{x^3 + 8}{x^2 + 2} = L \). Explain how this equality can be proved via the $\epsilon - \delta$ definition.

Exercise 2.1.14. Let $f(x) = \frac{1}{x}$.
(a) Use quantifiers to negate the statement “$\lim_{x \to 0} f(x)$ exists”.
(b) Prove that $\lim_{x \to 0} f(x)$ does not exist.

Exercise 2.1.15. Let $f(x) = \frac{1}{x}$.
(a) Find all real values $x_0$ for which $f(x)$ has a limit at $x_0$.
(b) Find all real values $x_0$ for which $f(x)$ is continuous at $x_0$.
(c) Prove both (b) and (c) using the $\epsilon - \delta$ definition.

Exercise 2.1.16. Use the $\epsilon - \delta$ definition to show that $\lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$.

Exercise 2.1.17. Let $f(x) = \frac{1}{x^2}$. Use the $\epsilon - \delta$ definition to show that $f$ is continuous on $\mathbb{R} \setminus \{0\}$.

Exercise 2.1.18. Let $f(x) = \sqrt{x}$. Use the $\epsilon - \delta$ definition to show that $f$ is continuous on $(0, \infty)$.

2.2. One sided limits

We want to make sense of the notion of limit of a function at a point $x_0$ in which we approach the point $x_0$ either from the right or from the left. This leads to the notion of one sided limits.

Definition 2.2.1. Let $x_0 \in \mathbb{R}$ and suppose that for some real number $\delta_0 > 0$, $f$ is defined on $(x_0, x_0 + \delta_0)$. Let also $L \in \mathbb{R}$. We say that the limit of $f(x)$ as $x$ approaches $x_0$ from the right equals $L$, and write

\[ \lim_{x \to x_0^+} f(x) = L, \]

if, for every $\epsilon > 0$, there exists $\delta \in (0, \delta_0]$ such that for all $x$ with $0 < |x - x_0| < \delta$ and $x > x_0$, it is true that $|f(x) - L| < \epsilon$.

If you compare Definition 2.2.1 with Definition 2.1.1, you will notice that the only change is in the condition on $x$. Specifically, we have replaced $0 < |x - x_0| < \delta$ with $0 < |x - x_0| < \delta$ and $x > x_0$. Equivalently, we could have written $0 < x - x_0 < \delta$.

Similarly, we have

Definition 2.2.2. Let $x_0 \in \mathbb{R}$ and suppose that for some real number $\delta_0 > 0$, $f$ is defined on $(x_0 - \delta_0, x_0)$. Let also $L \in \mathbb{R}$. We say that the limit of $f(x)$ as $x$ approaches $x_0$ from the left equals $L$, and write

\[ \lim_{x \to x_0^-} f(x) = L, \]

if, for every $\epsilon > 0$, there exists $\delta \in (0, \delta_0]$ such that for all $x$ with $0 < |x - x_0| < \delta$ and $x < x_0$, it is true that $|f(x) - L| < \epsilon$. 
In this case, we replaced $0 < |x - x_0| < \delta$ with $-\delta < x - x_0 < 0$.

**Question 2.2.3.** Suppose that for some number $L$, 
\[
\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L.
\]
What can be said about $\lim_{x \to x_0} f(x)$? Does the existence of the limit $\lim_{x \to x_0} f(x)$ say something about the two one-sided limits? What happens if the two one-sided limits are different?

**Definition 2.2.4.** We say that $f$ has a jump discontinuity at $x_0$ if both 
\[
\lim_{x \to x_0^+} f(x) \text{ and } \lim_{x \to x_0^-} f(x) \text{ exist, but } \lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x).
\]

**Definition 2.2.5.** We say that $f$ has a removable discontinuity at $x_0$ if both \(\lim_{x \to x_0^+} f(x)\) and \(\lim_{x \to x_0^-} f(x)\) exist, and \(\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L\), but either $f(x_0)$ is undefined or $L \neq f(x_0)$ if $f(x_0)$ is defined.

**Exercises**

In all exercises below you will need to use the precise definition of limit. Graphing the functions can give you an intuition of why the statement should be true, but you are expected to provide rigorous proofs.

**Exercise 2.2.6.** Let $f(x) = 4x + 7$ for $x < 1$ and $f(x) = -2x + 12.99$ for $x > 1$.
(a) Graph the function $f$.
(b) Find the two one-sided limits $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.
(c) What can be said about $\lim_{x \to 1} f(x)$?
(d) Is $f$ continuous at 1? Is $f$ continuous at 0 or at 2?

**Exercise 2.2.7.** Prove that $\lim_{x \to 0^+} \frac{|x|}{x} = 1$.

**Exercise 2.2.8.** Prove that $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

**Exercise 2.2.9.** Find all values of $x_0$ for which the two one-sided limits of $f(x) = |x|/x$ at $x_0$ are the same. Justify your claim rigorously using the $\epsilon - \delta$ definition of limit.

**Exercise 2.2.10.** Show that $\lim_{x \to 0^+} (x + \sqrt{x}) = 0$.

**Exercise 2.2.11.** Let $x_0 \in \mathbb{R}$. Give examples of functions $f$ and $g$ such that $f, g$ are both defined at $x_0$ and have jump discontinuities at $x_0$, but none of the functions $f + g$ and $f \cdot g$ have a jump discontinuity at $x_0$.

**Exercise 2.2.12.** The function $f$ is defined on $\mathbb{R}$, and its graph is made up only by arcs of parabolas. The function is not increasing nor decreasing on the whole real axis, and it is not always concave up or concave down.
Find a possible formula defining such a function \( f \) so that it has a jump discontinuity at a given point \( x_0 \).

### 2.3. Infinite limits at a point

Intuitively, we think of a *vertical asymptote* of a given function \( f \) as being a vertical line \( x = x_0 \) that is approached (but never touched) by the values \( f(x) \) when \( x \) approaches \( x_0 \). In other words, we expect that at least one of the two one-sided limits at \( x_0 \) is infinite, that is either \( \infty \) or \( -\infty \).

What do we mean by an infinite limit at a point?

**Definition 2.3.1.** Let \( x_0 \in \mathbb{R} \) and suppose that for some real number \( \delta_0 > 0 \), \( f \) is defined on \((x_0 - \delta_0, x_0) \cup (x_0, x_0 + \delta_0)\). We say that the limit of \( f(x) \) as \( x \) approaches \( x_0 \) is \( \infty \), and write

\[
\lim_{x \to x_0} f(x) = \infty,
\]

if, for every real number \( B \), there exists a \( \delta \in (0, \delta_0] \) such that for all \( 0 < |x - x_0| < \delta \), it is true that \( f(x) > B \).

Observe again that we do not care about the value of \( f \) at \( x_0 \); indeed, in many situations, this value is undefined.

**Question 2.3.2.** What are the precise definitions of

\[
\lim_{x \to x_0} f(x) = -\infty, \quad \lim_{x \to x_0^-} f(x) = \pm\infty, \quad \lim_{x \to x_0^+} f(x) = \pm\infty?
\]

**Example 2.3.3.** Show that \( \lim_{x \to 0} \frac{1}{x^2} = \infty \).

**Proof.** Let \( B \) be given. It is sufficient to consider \( B \geq 0 \) (why?). Choose \( \delta = B^{-1/2} > 0 \) and suppose that \( 0 < |x| < \delta \). Then

\[
\frac{1}{x^2} \geq \frac{1}{\delta^2} = \frac{1}{B^{-1}} = B.
\]

\( \square \)

**Question 2.3.4.** Can you find \( \delta \) directly by starting from the inequality \( f(x) > B \)?

**Exercises**

In most exercises below you will need to use the precise definition of limit. Graphing the functions can give you an intuition of why the statement should be true, but you are expected to provide rigorous proofs.

**Exercise 2.3.5.** Use quantifiers to re-write Definition 2.3.1. Use quantifiers to negate Definition 2.3.1.

**Exercise 2.3.6.** Use quantifiers to re-write and then negate each one of the definitions you stated in Question 2.3.2.
2.4. Limit theorems

While it is very instructive to prove a limit of a given function from scratch using the $\epsilon - \delta$ definition, we have the feeling that sometimes we might be doing just a bit too much. Perhaps, there is a shortcut that tells us how to compute the limit of an arithmetic combination of functions whose limits we already know. Of course, we would need to prove that this shortcut holds (strictly using the definition) and once this is done we could further use it in more complicated examples. Below, I will provide proofs of a couple of such shortcuts (theorems or laws about limits).

Assume that we have two functions $f$ and $g$ and each one of them has a limit at a given $x_0 \in \mathbb{R}$ (this $x_0$ could also be $\infty$ or $-\infty$, but in this case we would have to adapt our proofs accordingly; see Exercise 2.4.13). In particular, we assume that for some $\delta_0 > 0$, both $f$ and $g$ are defined on $(x_0 - \delta_0, x_0 + \delta_0) \setminus \{x_0\}$. Let

$$\lim_{x \to x_0} f(x) = L \in \mathbb{R} \quad \text{and} \quad \lim_{x \to x_0} g(x) = M \in \mathbb{R}.$$  

**Theorem 2.4.1.** (Sum Law) \(\lim_{x \to x_0} (f(x) + g(x)) = L + M\).

**Proof.** Let $\epsilon > 0$ be given. By Definition 2.1.1, we can choose $\delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}$$

and $\delta_2 > 0$ such that

$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2}.$$ 

Choose then $\delta = \min(\delta_0, \delta_1, \delta_2)$ and suppose that $0 < |x - x_0| < \delta$. We get

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

Exercise 2.3.7. Show that $\lim_{x \to 2} \frac{1}{|x - 2|} = \infty$.

Exercise 2.3.8. Show that $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Exercise 2.3.9. Show that $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

Exercise 2.3.10. Show that $\lim_{x \to 1} \frac{1}{(x - 1)^2} = \infty$.

Exercise 2.3.11. Show that $\lim_{x \to 0^+} \frac{x - 1}{x^2} = -\infty$.

Exercise 2.3.12. Show that $\lim_{x \to 0^-} \frac{x^2}{x - 1} = -\infty$.

Exercise 2.3.13. Show that $\lim_{x \to 0} |\csc x| = \infty$. 

2.4. Limit theorems
To write the first inequality, we used the triangle inequality, which is the topic of Exercise 1.2.8. Recall that for any two real numbers \(a, b\), we have \(|a + b| \leq |a| + |b|\). In the argument above, \(a = f(x) - L, b = g(x) - M\). This inequality is used repeatedly in proving (most) of the other limit theorems.

**Theorem 2.4.2.** (Product Law) \(\lim_{x \to x_0} (f(x)g(x)) = LM\).

**Proof.** Let \(\epsilon > 0\) be given. By Definition 2.1.1, we can choose \(\delta_1 > 0\) such that
\[
0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \min\left(\frac{\epsilon}{2(1 + |M|)}, 1\right)
\]
and \(\delta_2 > 0\) such that
\[
0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2(1 + |L|)}.
\]
Choose then \(\delta = \min(\delta_0, \delta_1, \delta_2) > 0\) and suppose that \(0 < |x - x_0| < \delta\). We have
\[
|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM|
\leq |M||f(x) - L| + |f(x)||g(x) - M|
< |M|\frac{\epsilon}{2(1 + |M|)} + (1 + |L|)\frac{\epsilon}{2(1 + |L|)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

\(\square\)

Similar ideas to the ones used in Theorems 2.4.1 and 2.4.2 allow us to prove the following limit theorems. The proofs of these theorems are left to you as an exercise. You are encouraged to try to prove them directly, that is, without appealing to a combination of other theorems that could imply the statement you are trying to prove.

**Theorem 2.4.3.** (Constant Law) If \(k\) is constant, then \(\lim_{x \to x_0} (kf(x)) = kL\).

**Theorem 2.4.4.** (Difference Law) \(\lim_{x \to x_0} (f(x) - g(x)) = L - M\).

Note that Theorem 2.4.3 follows directly from Theorem 2.4.2 as long as we assume Exercise 2.4.9, while Theorem 2.4.4 is a consequence of Theorem 2.4.1 and Theorem 2.4.3.

**Theorem 2.4.5.** (Reciprocal Law) If \(M \neq 0\), \(\lim_{x \to x_0} \frac{1}{g(x)} = \frac{1}{M}\).

**Theorem 2.4.6.** (Quotient Law) If \(M \neq 0\), \(\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}\).

**Theorem 2.4.7.** (Order Preserving Law) Assume that \(f(x) \leq g(x)\) on an open interval containing \(x_0\). Then, \(L \leq M\).
Exercises

Exercise 2.4.8. Show that for any \( x_0 \in \mathbb{R} \), \( \lim_{x \to x_0} x = x_0 \).

Exercise 2.4.9. Show that for any \( x_0 \in \mathbb{R} \) and any constant \( k \in \mathbb{R} \), \( \lim_{x \to x_0} k = k \).

Exercise 2.4.10. Which laws allow us to compute the value of \( \lim_{x \to 2} (x^3 - 2x^2 + x - 7) \)? Find the limit using these laws and the previous two exercises.

Exercise 2.4.11. Which laws allow us to compute the value of \( \lim_{x \to 2} \frac{x^2 - 4}{x + 4} \)? Find this limit.

Exercise 2.4.12. Suppose that \( f \) and \( g \) are continuous functions at \( x_0 \). Use the laws to prove that \( f \pm g, fg, kf \) are also continuous functions. Is this always true for \( f/g \)? Explain why or why not.

Exercise 2.4.13. Prove Theorems 2.4.1-2.4.6 for \( x_0 = \pm \infty \).

Exercise 2.4.14. Show that the limit of a function at a point is unique, that is, if \( \lim_{x \to x_0} f(x) = L \) and \( \lim_{x \to x_0} f(x) = l \), then necessarily \( L = l \).

Exercise 2.4.15. Prove that if \( \lim_{x \to x_0} f(x) = \infty \), then the statement \( \exists L \in \mathbb{R}, \lim_{x \to x_0} f(x) = L \) is false.

Exercise 2.4.16. Use Theorem 2.4.7 to prove the following Squeeze Law: If \( f(x) \leq g(x) \leq h(x) \) on an open interval containing \( x_0 \) and \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L \), then \( \lim_{x \to x_0} g(x) = L \).

Exercise 2.4.17. Prove that if \( \lim_{x \to x_0} f(x) = L \geq 0 \), then \( \lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L} \).

2.5. L’Hôpital’s rule

In this section we revisit a tool that you have used before in a “standard” calculus course: l’Hôpital’s rule. You will see shortly that this rule is simply an application of the main Limit Laws learned before. In these lecture notes, L’Hôpital’s rule is used later on, for example, to justify the calculation of certain “harder” limits of sequences in Chapter 3.

We begin by recalling the definition of the derivative at a point. Given a function \( f \) defined on an open interval containing \( x_0 \in \mathbb{R} \), we say that \( f \) is differentiable at \( x_0 \) if the limit

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\]

exists. If this is the case, we call the value of this limit \( f'(x_0) \) (which is some real number). We notice right away that a necessary condition for \( f \) to be
differentiable at \( x_0 \) is that \( f \) is continuous at \( x_0 \). Indeed, for \( x \neq x_0 \), we can write
\[
f(x) - f(x_0) = (x - x_0) \frac{f(x) - f(x_0)}{x - x_0}.
\]
Thus, assuming \( f'(x_0) \) exists and using the Sum, Constant and Product Laws, we see that
\[
\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (x - x_0) \frac{f(x) - f(x_0)}{x - x_0} = f(x_0) + 0 \cdot f'(x_0) = f(x_0),
\]
which is exactly what we wanted to prove.

The general “0/0” version of l'Hôpital's rule is the following.

**Theorem 2.5.1.** Let \( f \) and \( g \) be two functions and \( x_0 \in \mathbb{R} \) such that \( f(x_0) = g(x_0) = 0 \). Assume that \( f(x), g(x), f'(x), g'(x) \) are defined on the set \( I \setminus \{x_0\} \), where \( I \) is an open interval containing \( x_0 \). Then
\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.
\]

We only prove here a weaker version of Theorem 2.5.1, which we now state. You are encouraged to try proving the general version above on your own.

**Theorem 2.5.2.** Let \( f \) and \( g \) be two functions and \( x_0 \in \mathbb{R} \) such that \( f(x_0) = g(x_0) = 0 \). Assume that \( f'(x_0) \) and \( g'(x_0) \) exist and \( g'(x_0) \neq 0 \). Then
\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.
\]

Before proving Theorem 2.5.2, let us convince ourselves that it “makes sense”. The gist of differentiability at \( x_0 \) is that we can approximate the function \( f \) as in the Theorem 2.5.2 by the linear function
\[
f(x) + f'(x_0)(x - x_0) = f'(x_0)(x - x_0);
\]
see also Chapter 5 for more on this approximation. A similar statement holds for the function \( g \). Thus, for \( x \approx x_0 \), we would write
\[
\frac{f(x)}{g(x)} \approx \frac{f'(x_0)(x - x_0)}{g'(x_0)(x - x_0)} = \frac{f'(x_0)}{g'(x_0)}.
\]
The proof below makes this intuition rigorous.

**Proof.** Since \( f(x_0) = g(x_0) = 0 \), for all \( x \neq x_0 \) we can write
\[
\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{x - x_0} \cdot \frac{x - x_0}{g(x) - g(x_0)}.
\]
Thus, simply passing to the limit on both sides of this equality, recalling the definition of derivative and using the Quotient Law, we obtain

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}{\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}} = \frac{f'(x_0)}{g'(x_0)}.
\]
CHAPTER 3

Sequences

3.1. Definition of an infinite sequence

As we have already seen in the previous two chapters, the concept of infinity is crucial in many situations and making it rigorous requires a careful use of words on our part. While the previous lectures dealt with the “continuous” infinite (real-valued functions), this time we wish to discuss the concept of infinity from a “discrete” perspective (sequences). As we shall soon see, the two situations are strongly connected. Indeed, modulo some small changes, the concept of limit of a sequence and that of a function at infinity are essentially the same.

In order to introduce the concept of sequence, let us begin by remembering that \( \sqrt{2} \) is an irrational number and that it is represented by an infinite decimal. But what exactly do we mean by that? We usually rephrase it as an approximate statement such as \( \sqrt{2} \) is about 1.41421356. Indeed, 1.41421356 is just an approximation of \( \sqrt{2} \), since obviously \((1.41421356)^2 = 1.99999999329\), which is not equal to 2. So, the correct decimal representation of \( \sqrt{2} \) is more like 1.41421356237... where the dots represent infinitely more numbers. We could collect this information into saying that \( \sqrt{2} \) is represented by an infinite list of real numbers in a given order:

\[
1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, 7, ...
\]

Changing the order changes the sequence. Clearly, the number 4.114213562... does not represent \( \sqrt{2} \) anymore! To specify the order, we not only list the terms but we also index them:

\[
a_1, a_2, a_3, a_4, a_5, ..., a_n, ...
\]

In our case \( a_1 = 1, a_2 = 4, a_3 = 1, a_4 = 4, a_5 = 1 \) and so on. The index \( n \) specifies where the term \( a_n \) occurs in the list. Of course, it is a bit hard to find out what \( a_{2007} \) is in the sequence considered. But, we do have a rigorous way of defining the \( n \)th term: \( a_n \) is the final digit of the first largest decimal that squares to less than 2 and has \( n \) digits after the decimal point. So, here we have a simple sequence not given by a simple formula: the sequence defining \( \sqrt{2} \) is described by a simply stated rule, but there does not seem to be a simple way of finding its large terms.

**Question 3.1.1.** The sequence \( a_1, a_2, a_3, ... \) is defined as follows: \( a_1 = 2 \) and thereafter each number equals the previous number minus its cube root.
3. SEQUENCES

How do you express this rule as an algebraic relation? How many occurrences of the number 2 are in the expression of $a_{2007}$?

In many examples we describe a sequence by writing a formula that specifies its terms. For example, writing $a_n = 2n + 1, n \geq 1$ is the same as listing its terms 1, 3, 5, 7, 9, ..., $2n + 1, ...$. The inequality $n \geq 1$ describes where the index $n$ starts (at 1). We always assume throughout these chapter that the index of a sequence $n$ belongs to $\mathbb{N}$. Note that $b_n = 2n + 5, n \geq 1$ describes the sequence 7, 9, 11, ...; but, obviously, this sequence is also described by $a_n = 2n + 1, n \geq 3$. In general, the first index of a sequence can be any integer $n_0$, and this convention is useful in certain calculations.

Any of the following notations is used to denote the sequence $a_1, a_2, ..., a_n, ...$:

$$(a_n)_{n \geq 1}, (a_n)_{n=1}^\infty, \{a_n\}_{n \geq 1}, \{a_n\}_{n=1}^\infty, \text{ or } a_n, n \geq 1.$$  

When the starting index is self-understood (and commonly taken as 1), we can just write $(a_n)$ or $\{a_n\}$ or $a_n$.

It is also useful sometimes to graph a sequence: either by marking the values of the terms on the real axis or by plotting points in the plane located at $(1, a_1), (2, a_2)$ and so on. This last representation leads in fact to the formal definition of an infinite sequence:

**Definition 3.1.2.** An infinite sequence of real numbers $(a_n)_{n \geq 1}$ is a function $f : \mathbb{N} \to \mathbb{R}$.

**Example 3.1.3.** The sequence 1, 3, 5, 7, ... can be thought of as the function that sends 1 to $a_1 = 1$, 2 to $a_2 = 3$, 3 to $a_3 = 5$, and so on. The graph of the sequence will consist of a set of discrete points in the plane: $(1, 1), (2, 3), (3, 5)$ and so on. All these points sit on the graph of $f(n) = 2n + 1$ defined on the set of positive integers. You should contrast the graph of $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = 2n + 1$ with the graph of $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with $\tilde{f}(x) = 2x + 1$.

**Question 3.1.4.** What is the relationship between the two functions $f$ and $\tilde{f}$?

**Exercises**

**Exercise 3.1.5.** Find a formula for the general term $a_n$ of the sequence 1, $-1$, 1, $-1$, 1, $-1$, ... Show two graphical representations of the sequence $a_n$ by considering the first 6 terms.  

**Exercise 3.1.6.** List the first 6 terms of the sequence defined by $b_n = \frac{1}{n}$. Show two graphical representations of the sequence $b_n$ by considering the first 6 terms.

**Exercise 3.1.7.** Compare the graphical representations of the sequences considered in the previous two exercises. Is the behavior of the two sequences similar? What essential difference do you notice?
3.2. CONVERGENCE

Exercise 3.1.8. Can you find a pattern in the finite sequence
\(-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}\)?
Assuming that this pattern continues indefinitely and gives an infinite sequence, what is a formula for its general term \(c_n\)? What common features and what differences do sequences \(b_n\) (defined above) and \(c_n\) have?

Exercise 3.1.9. The sequence \((a_n)\) is defined recursively by \(a_1 = 0, a_{n+1} = a_n + 1\). Find the general formula of \(a_n\).

Exercise 3.1.10. The sequence \((a_n)\) is defined recursively by \(a_1 = 1, a_{n+1} = a_n + n + 1\). Find the general formula of \(a_n\).

Exercise 3.1.11. The Fibonacci sequence is defined recursively by \(F_1 = F_2 = 1\) and \(F_{n+2} = F_{n+1} + F_n, n \geq 1\). Find \(F_{10}\). Explain in your own words what strategy you would take to find \(F_{2007}\) and whether you think this strategy is convenient.

Exercise 3.1.12. Let \(\varphi = (1 + \sqrt{5})/2\) denote the so-called golden ratio.
(a) Show that \(\varphi(\varphi - 1) = 1\).
(b) Show that any combination of the form \(a\varphi^n + b(1 - \varphi)^n\), where \(a, b \in \mathbb{R}\), verifies the recursion \(F_{n+2} = F_{n+1} + F_n, n \geq 1\).

Exercise 3.1.13. Use the previous exercise to find the general formula of the Fibonacci sequence \(F_n\) defined in Exercise 3.1.11.

Exercise 3.1.14. The Lucas sequence is defined recursively by \(L_1 = 2, L_2 = 1\) and \(L_{n+2} = L_{n+1} + L_n, n \geq 1\). Find the general formula of \(L_n\).

Exercise 3.1.15. Let \(F_n\) and \(L_n\) denote the Fibonacci respectively Lucas sequences defined above. Prove the following identities:
(a) \(L_n = F_{n-1} + F_{n+1}\);
(b) \(F_n = (L_{n-1} + L_{n+1})/5\);
(c) \(L_n^2 = 5F_n^2 + 4(-1)^n\).

3.2. Convergence

Consider the sequence \(a_n = \frac{n-1}{n}, n \geq 1\). Simply listing a few terms in the sequence, it is easy to see that as the index \(n\) increases and becomes larger and larger, the terms \(a_n\) get closer and closer to 1. This behavior should remind you of something that we have already discussed in the Chapter 1: a function having a finite limit at infinity! Indeed, recalling that a sequence is itself a function (having the positive integers as its domain), the precise definition of a limit of a sequence will mimic the definition you have previously encountered.

Definition 3.2.1. We say that a sequence \((a_n)\) converges to the real number \(L\) and write
\[
\lim_{n \to \infty} a_n = L \text{ (or } a_n \to L \text{ as } n \to \infty)\]
if for every $\epsilon > 0$ there exists an $N \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $n > N$, we have $|a_n - L| < \epsilon$. The number $L$ is called the \textit{limit} of the sequence.

In terms of the learned quantifiers, we can express our definition as follows:

$$\lim_{{n \to \infty}} a_n = L \iff \forall \epsilon > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N} \land n > N, |a_n - L| < \epsilon.$$ 

\textbf{Question 3.2.2.} In general, what quantity (quantities) does the $N$ in the definition of convergence depend on?

You may also wonder: what happens if we consider the limit of a sequence at a point, say at 1, $\lim_{{n \to 1}} a_n$? As it turns out, this is not very interesting. Indeed, having positive integers $n$ arbitrarily close to another integer (for example 1), means that $n$ equals that integer ($n = 1$). Thus, the only interesting behavior of discrete lists has to do with the index becoming increasingly larger!

\textbf{Example 3.2.3.} Show that the sequence $a_n = 1/n^3$ converges to 0.

\textbf{Proof.} Let $\epsilon > 0$ be given. Let $N \in \mathbb{R}$ be such that $N \geq 1/\sqrt[3]{\epsilon}$. Assume that $n \in \mathbb{N}$ and $n > N$. Then

$$|a_n - 0| = \frac{1}{n^3} < \frac{1}{N^3} \leq \epsilon.$$ 

\hfill $\square$

\textbf{Question 3.2.4.} How do we know what the choice of $N$ should be? Is there only one choice of $N$ that we can make? What is the smallest possible choice of a positive integer $N$ that will make the argument work?


\textbf{Proof.} In this case $a_n = 2007$ for all $n \geq 1$. Let $\epsilon > 0$ be given. Let $N$ be any positive real. Assume that $n \in \mathbb{N}$ is such that $n > N$. Then

$$|a_n - 2007| = 0 < \epsilon.$$ 

\hfill $\square$

\textbf{Example 3.2.6.} Show that $a_n = \sqrt{1 - \frac{1}{n}}$ converges to 1.

\textbf{Proof.} Let $\epsilon > 0$ be given. Let $N \in \mathbb{R}$ be such that $N \geq 1/\epsilon$. Assume that $n \in \mathbb{N}$ and $n > N$. Then

$$|a_n - 1| = \left| \frac{(1 - \frac{1}{n}) - 1}{1 + \sqrt{1 - \frac{1}{n}}} \right| \leq \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$ 

\hfill $\square$
3.2. CONVERGENCE

Here is an alternate proof.

**Proof.** Let $0 < \epsilon < 1$ be given (why is this assumption not restrictive?). Let $N$ be such that $N \geq 1/(1 - (1 - \epsilon)^2)$. Assume that $n \in \mathbb{N}$ and $n > N$. Then

$$a_n < 1 < 1 + \epsilon$$

and

$$a_n = \sqrt{1 - \frac{1}{n}} > \sqrt{1 - \frac{1}{N}} \geq 1 - \epsilon.$$

The two inequalities are equivalent to $|a_n - 1| < \epsilon$.

As you might have noticed, in the first proof of the last example we used an algebraic trick that allowed us to reduce the problem to that of comparing the simpler expression $1/n$ with the prescribed error. It is of course natural to ask if there is a shortcut to this proof via some properties of limits. These limit laws do exist and are similar to the ones proved in Chapter 2. As we shall soon see, they can be useful when trying to compute limits of sequences that have more complicated expressions. We postpone the issue of computing limits using these limit laws for the last section of this chapter.

**Exercises**

In all exercises below you will need to use the precise definition of convergence of a sequence. Graphing the sequences can give you an intuition of why the statement should be true, but you are expected to provide rigorous proofs along the lines outlined in the Examples 3.2.3-3.2.6.

**Exercise 3.2.7.** Show that $\lim_{n \to \infty} \frac{e^{n+1} - 1}{e^n} = e$.

**Exercise 3.2.8.** Show that the sequence $1/2, -1/5, 1/8, -1/11, 1/14, ...$ converges to 0.

**Exercise 3.2.9.** Does the sequence $\left(\frac{\cos n}{n}\right)$ converge? If yes, find its limit and justify your answer rigorously using the $\epsilon$–definition of convergence.

**Exercise 3.2.10.** Given a real number $x$, $[x]$ denotes the largest integer less or equal than $x$. Investigate the convergence of the sequence $[1 - 1/n]$.

**Exercise 3.2.11.** Show that if $a$ is a real number such that $|a| < 1$, then $\lim_{n \to \infty} a^n = 0$. Discuss the case $|a| = 1$.

**Exercise 3.2.12.** Prove that $\lim_{n \to \infty} \frac{n^2 + n}{2n^2 + 1} = \frac{1}{2}$.

**Exercise 3.2.13.** Prove that $\lim_{n \to \infty} \sin \left(\frac{1}{n}\right) = 0$. 
EXERCISE 3.2.14. Let \((a_n)\) and \((b_n)\) be two sequences that are the same with the exception of a finite number of terms. Show that if \((a_n)\) converges, then \((b_n)\) also converges and the limits are the same.

EXERCISE 3.2.15. Show that if a sequence is convergent then its limit is unique. Compare to the statement about the limit of a function in Exercise 2.4.14.

EXERCISE 3.2.16. Show that any convergent sequence \((a_n)\) is a Cauchy sequence, that is:

\[
\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall m, n > N, |a_m - a_n| < \epsilon.
\]

EXERCISE 3.2.17. Show that if \((a_n)\) is a convergent sequence then for, any fixed index \(p\), the sequence \((a_{n+p})\) is also convergent.

### 3.3. Divergence

**Definition 3.3.1.** The sequence \((a_n)\) is called divergent if the negation of Definition 3.2.1 is true.

In terms of the learned quantifiers, we can express the fact that \((a_n)\) diverges as follows:

\[
\forall L \in \mathbb{R}, \exists \epsilon > 0, \forall N \in \mathbb{R}, \exists n > N, |a_n - L| \geq \epsilon.
\]

**Example 3.3.2.** The sequence \(a_n = (-1)^n\) diverges.

**Proof.** We proceed by way of contradiction. Assume that \(a_n \to L\) as \(n \to \infty\), where \(L\) is some real number. Let \(\epsilon = 0.7\). By Definition 3.2.1, there exists some \(N \in \mathbb{R}\) such that for all \(n > N\) we have \(|a_n - L| < 0.7\). In particular, we can select an even number \(n_1\) and an odd number \(n_2\) (both larger than \(N\)) such that

\[
|a_{n_1} - L| = |1 - L| < 0.7
\]

and

\[
|a_{n_2} - L| = |-1 - L| < 0.7.
\]

Let us inspect more closely the two inequalities: the first one tells us that \(0.3 < L < 1.7\), while the second \(-1.7 < L < -0.3\) and this is obviously a contradiction. \(\square\)

**Question 3.3.3.** What other choices of \(\epsilon > 0\) would make the previous argument work?

**Example 3.3.4.** The sequence \(1, 1, 2, 2, -1, -1, -2, -2, 1, 1, 2, 2, -1, -1, -2, -2, ...\) diverges.

**Example 3.3.5.** The sequence \(1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ...\) diverges.
There is a difference, however, between the way the sequences in Examples 3.3.2 and 3.3.4 diverge from the sequence considered in Example 3.3.5. While in the first two examples the terms in the sequences do not increase in magnitude by more than 2, no matter how large the index \( n \) is, in the last example the terms become larger and larger than any fixed number we choose as the index \( n \) increases. This is again reminiscent of a notion learned previously for functions: infinite limit at infinity! As we have done there, we can precisely capture this notion for sequences.

**Definition 3.3.6.** We say that the sequence \((a_n)\) diverges to infinity and we write

\[
\lim_{n \to \infty} a_n = \infty \quad \text{(or } a_n \to \infty \text{ as } n \to \infty)\]

if for every real \( B \) there exists a number \( N \) such that for all \( n \in \mathbb{N} \) with \( n > N \), \( a_n > B \) is true.

**Question 3.3.7.** What further assumption can you make about \( B \) that will not affect Definition 3.3.6?

**Definition 3.3.8.** We say that the sequence \((a_n)\) diverges to negative infinity and we write

\[
\lim_{n \to \infty} a_n = -\infty \quad \text{(or } a_n \to -\infty \text{ as } n \to \infty)\]

if for every real \( B \) there exists a number \( N \) such that for all \( n \in \mathbb{N} \) with \( n > N \), \( a_n < B \) is true.

**Question 3.3.9.** What further assumption can you make about \( B \) that will not affect Definition 3.3.8?

**Example 3.3.10.** Show that \( \lim_{n \to \infty} \sqrt[n]{n} = \infty \).

**Proof.** Let \( B > 0 \) be given. Let \( N \) be such that \( N \geq B^3 \). Assume that \( n > N \). Then

\[
a_n = \lim_{n \to \infty} \sqrt[n]{n} > \lim_{n \to \infty} \sqrt[n]{N} \geq B.
\]

**Question 3.3.11.** What is the smallest choice of a positive integer \( N \) for which the argument works?

**Example 3.3.12.** The sequence \( a_n = (-1)^n \) does not diverge to infinity (or negative infinity).

**Proof.** Choose \( B = 2 \). Let \( N \) be any positive real. There exists \( n = [N] + 1 > N \) such that \( a_n < B \). For negative infinity, choose \( B = -2 \). □

**Exercises**
In all exercises below you will need to use the precise definition of divergence of a sequence. Graphing the sequences can give you an intuition of why the statement should be true, but you are expected to provide rigorous proofs along the lines outlined in the Examples 3.3.10-3.3.12.

**Exercise 3.3.13.** Use quantifiers to restate Definitions 3.3.6 and 3.3.8.

**Exercise 3.3.14.** Use quantifiers to negate Definitions 3.3.6 and 3.3.8.

**Exercise 3.3.15.** Show that the sequence in Example 3.3.4 does not diverge to infinity (or negative infinity).

**Exercise 3.3.16.** Find a formula for the $n$th term of the sequence in Example 3.3.5. Show that this sequence diverges to infinity.

**Exercise 3.3.17.** Show that the sequence $a_n = \sqrt{n} - n$ diverges to negative infinity.

**Exercise 3.3.18.** Let $F_n$ be the Fibonacci sequence. Prove that $F_n$ diverges to infinity.

**Exercise 3.3.19.** Use Exercise 3.1.15 (a) to prove that the Lucas sequence $L_n$ diverges to infinity.

**Exercise 3.3.20.** Use Exercise 3.1.15 (c) to compute $\lim_{n \to \infty} \left( \frac{L_n}{F_n} \right)^2$.

**Exercise 3.3.21.** Compute $\lim_{n \to \infty} \frac{F_{2n}}{F_n}$.

### 3.4. Limit theorems. Computing the limit of a sequence

Since a sequence is just a function defined on the set of positive integers, it is not surprising that the usual limit laws that we have encountered in Chapter 2 also apply to sequences.

**Theorem 3.4.1.** Let $(a_n)$ and $(b_n)$ be two sequences that converge, and let $\lim_{n \to \infty} a_n = L$, $\lim_{n \to \infty} b_n = M$. Then

1. $\lim_{n \to \infty} (a_n \pm b_n) = L \pm M$;
2. $\lim_{n \to \infty} (a_n b_n) = LM$;
3. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.

The proof of this theorem is left to you as an exercise. You simply have to adapt the arguments outlined in Chapter 2, Section 4. As an illustration, let us look at the following example.

**Example 3.4.2.** Find the limit of the sequence $\left( \frac{n^2 + 1}{2n^2 - 3} \right)_{n \geq 1}$.
Proof. Rewrite the sequence as
\[ \frac{1 + \frac{1}{n^2}}{2 - 3\frac{1}{n^2}}. \]
If we now use Theorem 3.4.1, we conclude that the limit is \( \frac{1}{2} \). \( \square \)

Recall now that limits of functions preserve order; see Theorem 2.4.7 in Chapter 2, Section 4. This property further implies the so-called Squeeze Law; see Exercise 2.4.16. The same holds in the discrete case. The order preserving property of limits of sequences is deferred to the exercises section (see Exercise 3.4.29). We now state and prove the Squeeze Law for sequences. If you have already solved Exercise 2.4.16, you will notice that the proofs are essentially the same.

**Theorem 3.4.3.** (Squeeze Law) Let \((a_n)\) and \((b_n)\) be two sequences that converge to the same limit \(L\). If \((c_n)\) is some other sequence such that \(a_n \leq c_n \leq b_n\) for all \(n \geq n_0\), where \(n_0 \in \mathbb{N}\) is some fixed index, then \((c_n)\) also converges to \(L\).

Proof. Let \(\epsilon > 0\) be given. There exist \(N_1\) and \(N_2\) such that
\[ L - \epsilon < a_n < L + \epsilon, \text{ for all } n > N_1, \]
\[ L - \epsilon < b_n < L + \epsilon, \text{ for all } n > N_2. \]
Then, the assumption on \(c_n\) implies that, for \(n > N = \max(N_1, N_2, n_0)\), we also have \(L - \epsilon < c_n < L + \epsilon\). \( \square \)

Here are two immediate consequences of Theorem 3.4.3:

1. \(|a_n| \leq b_n\) and \(b_n \to 0 \Rightarrow a_n \to 0;\)
2. \(|a_n| \to 0 \Rightarrow a_n \to 0.\)

**Example 3.4.4.** Show that \(\lim_{n \to \infty} \frac{n!}{n^n} = 0.\)

Proof. Recall that \(n! = 1 \cdot 2 \cdots n.\) Clearly
\[ 0 < \frac{n!}{n^n} < \frac{1}{n} \]
and we can apply now the Squeeze Law with \(a_n = 0, b_n = 1/n\) and \(c_n = n! / n^n.\) \( \square \)

**Example 3.4.5.** Show that \(\lim_{n \to \infty} \frac{2^n}{n!} = 0.\)

Proof. We have
\[ 0 < \frac{2^n}{n!} < \frac{4}{n} \]
and again we can apply the Squeeze Law. \( \square \)
Another important and very powerful tool that we can borrow from limits of functions is l’Hôpital’s rule. We recall that this rule deals with so-called indeterminate cases such as $0/0$ or $\infty/\infty$. The following result will show us why we are entitled to use the rule in the discrete case as well.

**Theorem 3.4.6.** Let $a_n = f(n)$, where $f(x)$ is a real-valued function. If $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} a_n = L$.

**Proof.** Let $\epsilon > 0$ be given. By the definition of a finite limit at infinity (see Chapter 1), there exists an $A \in \mathbb{R}$ such that for all $x > A$, $|f(x) - L| < \epsilon$. Assume that $n > A$. Then

$$|a_n - L| = |f(n) - L| < \epsilon,$$

and we are done. \qed

**Remark 3.4.7.** The converse of this theorem is false. We can easily see this by considering, for example, the function $f(x) = \cos(2\pi x)$. Remember that this function does not have a limit at infinity (see Exercise 1.3.8 in Chapter 1, Section 3). But, clearly

$$a_n = f(n) = \cos(2\pi n) = 1 \to 1.$$

**Example 3.4.8.** (Example 3.4.2 revisited) We note that we fall under the scope of Theorem 3.4.6 with

$$\lim_{n \to \infty} \frac{n^2 + 1}{2n^2 - 3} = \frac{1}{2}.$$

The last equality follows from one application of l’Hôpital’s rule. In fact, knowing that Theorem 3.4.6 holds, we are allowed to be a bit sloppy and write the following sequence of equalities:

$$\lim_{n \to \infty} \frac{n^2 + 1}{2n^2 - 3} = \lim_{n \to \infty} \frac{2n}{4n} = \frac{1}{2}.$$

**Example 3.4.9.** Show that $\lim_{n \to \infty} \frac{\ln n}{n} = 0$.

**Proof.** We apply l’Hôpital’s rule to

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

Again, because of Theorem 3.4.6, we can be a bit sloppy and simply write:

$$\lim_{n \to \infty} \frac{\ln n}{n} = \lim_{n \to \infty} \frac{1/n}{1} = 0.$$

**Example 3.4.10.** Show that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$. 

3.4. LIMIT THEOREMS. COMPUTING THE LIMIT OF A SEQUENCE

Proof. This is a limit in the indeterminate form $1^\infty$. We try to change it into a form for which l'Hôpital's rule works, specifically $0/0$. We do so by first rewriting the sequence $a_n$ as $e^{\ln a_n}$. The details are as follows:

$$\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = \lim_{n \to \infty} e^{n \ln(1+\frac{1}{n})} = e^{\lim_{n \to \infty} \frac{\ln(1+\frac{1}{n})}{1/n}}. $$

The second equality is justified by the statement of Exercise 3.4.30. The limit in the exponential is now in the $0/0$ form. We can go ahead and apply the rule there to obtain

$$\lim_{n \to \infty} \frac{\ln(1+\frac{1}{n})}{1/n} = \lim_{n \to \infty} -\frac{1}{n^2} \cdot \frac{1/(1+1/n)}{-1/n^2} = 1.$$

Therefore tracing back and taking the exponential we arrive to our desired limit.

The purpose of the last three examples is to encourage you to review l'Hôpital's rule or other techniques for computing limits that you have previously learned for functions. Once you have reviewed these techniques, go ahead and try your hand on exercises.

We finish our tour of limit theorems with another powerful result which allows us to prove the convergence of a sequence indirectly. The significance of this result will become clearer once we delve into the study of infinite series. Some definitions are in order.

Definition 3.4.11. A sequence $(a_n)$ is called non-decreasing (or non-increasing) if $a_n \leq a_{n+1}$ (or $a_n \geq a_{n+1}$) for all $n$. A sequence is called monotonic if it is either non-increasing or non-decreasing.

Definition 3.4.12. We say that the sequence $(a_n)$ is bounded above if there is a number $M \in \mathbb{R}$ (called an upper bound) such that $a_n \leq M$ for all $n$. The sequence $(a_n)$ is bounded below if there is a number $m \in \mathbb{R}$ (called a lower bound) such that $a_n \geq m$ for all $n$. We call a sequence bounded if it is bounded both above and below.

Question 3.4.13. Why is it true that $(a_n)$ being bounded is equivalent to the existence of a number $K > 0$ such that $|a_n| \leq K$ for all $n$?

It is easy to prove that any convergent sequence is necessarily bounded (Exercise 3.4.24). While the converse of this statement is not true (consider, for example, the sequence $(-1)^n$), by adding the assumption of monotonicity we do get a true statement.

Theorem 3.4.14. A bounded and monotonic sequence is convergent.

The proof of this theorem uses an important assumption about the system of real numbers, the so-called completeness axiom. Its proof is beyond the scope of this lecture notes.
QUESTION 3.4.15. Explain why Theorem 3.4.14 is equivalent to any of the following two seemingly weaker statements:

(1) Any non-decreasing and bounded above sequence is convergent.
(2) Any non-increasing and bounded below sequence is convergent.

EXAMPLE 3.4.16. (Exercise 3.4.5 revisited) Analyze the convergence of the sequence \( a_n = \frac{2^n}{n!} \).

PROOF. Note that the question makes no reference to the exact value of the limit. Clearly, all the terms are positive. Then

\[
\frac{a_{n+1}}{a_n} = \frac{2}{n + 1} \leq 1, \forall n \geq 1.
\]

This proves that the sequence is non-increasing. In particular, for all \( n \) we have \( 0 < a_n \leq a_1 = 2 \), which proves that it is also bounded. By Theorem 3.4.14, the sequence is convergent. To find the limit of the sequence we could use one of the properties previously learned (in Theorem 3.4.1). Indeed, if we let \( \lim_{n \to \infty} a_n = L \), then

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \left( a_n \frac{2}{n + 1} \right) = \left( \lim_{n \to \infty} a_n \right) \cdot \left( \lim_{n \to \infty} \frac{2}{n + 1} \right) = L \cdot 0 = 0.
\]

Note that in order to use this trick we needed to know \textit{a priori} that \( L \) exists and is finite! \qed

EXAMPLE 3.4.17. The sequence \( a_n = \sqrt{n} \) is non-decreasing but not bounded.

EXAMPLE 3.4.18. The sequence \( a_n = \frac{n}{n + 1} \) is non-decreasing and bounded.

PROOF. Look at \( a_{n+1} - a_n = \frac{1}{(n+1)(n+2)} > 0 \). Clearly \( 0 < a_n < 1 \). The sequence converges by Theorem 3.4.14. To find its limit, however, we cannot use the trick in Example 3.4.16, but we can use, for example, L'Hôpital's rule and find the limit to be 1. \qed

EXAMPLE 3.4.19. The sequence \( a_n = (-1)^n \) is bounded but non-monotonic.

There are many ways of deciding the behavior of a sequence. Which approach one takes depends on the specific sequence considered. Practicing several exercises will help you get used to the variety of properties and techniques learned, which in turn will help you decide what is the best way to treat a particular case. There is no recipe really, but having a small dictionary of frequently appearing sequences definitely helps.

Exercises

EXERCISE 3.4.20. Prove Theorem 3.4.1.
EXERCISE 3.4.21. Prove the two consequences stated immediately after the proof of Theorem 3.4.3.

EXERCISE 3.4.22. Use Theorem 3.4.3 to show that \( \lim_{n \to \infty} \frac{(-1)^n \ln n}{\sqrt{n}} = 0 \).

EXERCISE 3.4.23. Determine whether each of the sequences below converges or diverges. If it converges, find its limit. Explain with sufficient details each claim.

1. \( a_n = \frac{n - 2}{2n + 7} \);
2. \( a_n = \frac{(-1)^n(n - 2)}{2n + 7} \);
3. \( a_n = \frac{n - 2}{2n^2 + 7} \);
4. \( a_n = n^2e^{-n} \);
5. \( a_n = n^2(0.5)^n \);
6. \( a_n = (n\pi + \pi/2) \);
7. \( a_n = n^{1/n} \);
8. \( a_n = \left(1 - \frac{2}{n}\right)^n \);
9. \( a_n = \sqrt{n^2 + 1} - n \);
10. \( a_n = n \sin \frac{1}{n} \).

EXERCISE 3.4.24. Prove that any convergent sequence is bounded.

EXERCISE 3.4.25. Show that Theorem 3.4.14 can be used to investigate the convergence of the following sequences. When possible, find their limits.

1. \( a_n = (0.5)^n \);
2. \( a_n = \frac{2^n}{(n!)^3} \);
3. \( a_n = n(0.5)^n \);
4. \( a_n = \frac{n - 2}{n + 2} \);
5. \( a_n = \frac{n^n}{n!} \), where \( a \) is a fixed real number.

EXERCISE 3.4.26. Can we use Theorem 3.4.14 to determine the convergence of the sequence \( a_n = (\cos n)/n \)? Can we find the limit of this sequence?

EXERCISE 3.4.27. The sequence \( a_n \) is defined recursively by \( a_1 = 2, a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \), \( n \geq 1 \).

(a) Show that \( a_n^2 - 2 = \left(\frac{a_{n-1}}{2} - \frac{1}{a_{n-1}}\right)^2 \);
(b) Is \( a_n \) bounded?
(c) Show that \( a_n \) is non-increasing.
(d) Show that \( a_n \) is convergent and find its limit.
Exercises 3.4.28. The sequence $a_n$ is defined recursively by $a_1 = 1/2$, $a_{n+1} = \max\{a_n, \cos n\}$. Show that $a_n$ is convergent. Is it easy to find its limit? How would we "guess" its limit?

Exercises 3.4.29. Show the following Order Preserving Law for sequences: If $a_n \leq b_n$ for all $n$, and $a_n \to L$, $b_n \to M$ as $n \to \infty$, then $L \leq M$. Then, use this result to prove that, if a sequence is convergent, its limit is unique.

Exercises 3.4.30. Prove the following Continuity Law for sequences: If $a_n \to L$ as $n \to \infty$ and $f$ is a real valued function continuous at $L$, then $b_n = f(a_n) \to f(L)$ as $n \to \infty$.

Exercises 3.4.31. Use Exercise 3.4.30 to give a different proof of Example 3.2.6.

Exercises 3.4.32. Use Exercise 3.4.30 to find the limit of the sequence $b_n = 2^{1/n}$.

Exercises 3.4.33. Use Exercises 3.4.30 and 3.3.20 to find the limit of the sequence $F_n/L_n$, where $F_n$ and $L_n$ denote the Fibonacci and Lucas sequences, respectively.
CHAPTER 4

Infinite series

4.1. Definition of infinite series

Assuming the familiar rules of arithmetic, we know that as long as we have two real numbers \( a, b \) such that their average equals either \( a \) or \( b \), then the two numbers are equal to each other (\( a = b \)). Indeed, if we assume for example that \( (a + b)/2 = a \), then \( (a - b)/2 = 0 \), or \( a - b = 0 \), which in turn implies that \( a = b \).

Following our reasoning above let us look at \( a = 1 \) and \( b = 0.99999... \), the dots meaning that we have infinitely many nines after the decimal point. Now, if you look at the average of the two numbers, you get

\[
\frac{a + b}{2} = \frac{1.99999...}{2} = 0.99999... \neq b,
\]

so \( a = b \) or \( 1 = 0.99999... \)?! If you do not like the fact that we work with infinitely many decimals, then change your point of view and think of the equality \( 1 = 0.99999... \) in terms of approximations: no matter what error margin \( \epsilon > 0 \) you pick, you can make the difference between the two numbers considered smaller than \( \epsilon \). For example, if \( \epsilon = 0.001 \), since \( 0.99999... > 0.999 \), we have \( 1 - 0.99999... < 1 - 0.999 = 0.001 \).

Here is another useful way of looking at the number \( 0.99999... \) (and its meaning). If we truncate the number after one decimal, two decimals, etc, we get a sequence of truncated numbers representing partial sums. Each truncations is obtained from the previous one by adding to it an extra 9. We can write

\[
s_1 = 0.9 = \frac{9}{10},
\]

\[
s_2 = 0.99 = \frac{9}{10} + \frac{9}{10^2} = s_1 + \frac{9}{10^2},
\]

\[
s_3 = 0.999 = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} = s_2 + \frac{9}{10^3},
\]

and so on, the \( n \)th truncation being defined in terms of the previous one by

\[
s_n = 0. \underbrace{99...9}_n = s_{n-1} + \frac{9}{10^n}.
\]

If we continue this way we eventually would want to write

\[
0.99999... = 9(0.1) + 9(0.1)^2 + 9(0.1)^3 + \cdots 9(0.1)^n + \cdots = \sum_{n=1}^{\infty} 9(0.1)^n.
\]
Of course, adding finitely many terms creates no problems, but what about making sense of an infinite sum? Well, one can almost sense now that to justify rigorously the equality $1 = 0.99999...$ one needs to look at the sequence of truncations of the number $0.99999...$ or, as we will call it from now on, the sequence of partial sums, and show that it converges to 1. Intuitively, this is clear: if we add more and more terms, the sum gets closer and closer to 1.

The issue of trying to add infinitely many numbers is not new. It goes back to the ancient Greeks, in particular Zeno of Elea and one of his paradoxes of motion (designed to prove that motion is just an illusion). Imagine the race between the fast running Achilles and a very slow turtle. Suppose that the turtle has a constant speed of 1 ft/s and that Achilles runs ten times faster than the turtle (10 ft/s), but that the turtle starts 10 ft ahead of the Greek hero. Now, we know that Achilles will catch up eventually and then overtake the slow turtle. But Zeno argued that Achilles can never overtake the turtle. This is a paradox! Here is his argument: after 1s Achilles covers the 10 ft head-start that the turtle had, but the turtle is now 1 ft ahead. After 0.1s, Achilles covers the 1 ft, but the turtle is now 0.1 ft ahead. In another 0.01s, Achilles covers the 0.1 ft ahead of him, but the turtle is still 0.01 ft ahead. Thus whenever Achilles reaches somewhere the turtle has been, he still has to go further...and thus he will never catch up! The solution of course is easy: it does not necessarily take an infinite amount of time to traverse an infinite sequence of distances that become (suitably) smaller and smaller. In fact, Achilles will overtake the turtle at the 11.1111... ft mark, which equals the infinite sum $10 + 1 + 0.1 + 0.01 + \cdots$. Can we actually add the terms one by one? No! But we can make sense of their sum though a limiting process.

**Question 4.1.1.** What rational number is 11.1111... equal to?

In general, we want to make sense of an infinite series of real numbers, that is the sum of an infinite sequence $(a_n)_{n \geq 1}$ of numbers:

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n.$$

As for sequences, we can replace the index $n = 1$ with another index $n = n_0$, where $n_0$ is some fixed positive integer.

**Definition 4.1.2.** For a given sequence $(a_n)_{n \geq 1}$, we define the sequence of partial sums $s_1, s_2, ..., s_n, ...$ by $s_1 = a_1, s_n = s_{n-1} + a_n$. We say that the series $\sum_{n \geq 1} a_n$ converges to $s$ and write

$$\sum_{n=1}^{\infty} a_n = s.$$
if the sequence of partial sums converges to $s$, that is $s_n \to s$ as $n \to \infty$. In this case we call $s$ the sum of the series. We say that the series diverges if the sequence of partial sums diverges.

When the summation is understood, we will simply write $\sum_n a_n$ or $\sum a_n$. With some abuse of notation, it is common to refer to a series and its sum the same way. In what follows, we will try to be consistent in our notation and use the notation $\sum_{n=1}^{\infty}$ to denote only the sum of a series. Changing the index $n$ to $k$ is of no importance, the sums

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{k=1}^{\infty} a_k$$

are the same.

**Example 4.1.3.** The series $0 + 0 + 0 + \cdots$ converges to 0.

**Proof.** Clearly, $s_n = 0$ for all $n$, therefore $s_n \to 0$ as $n \to \infty$. \qed

**Example 4.1.4.** The series $1 + 1 + 1 + \cdots$ diverges to infinity.

**Proof.** Clearly, $s_n = n \to \infty$ as $n \to \infty$. \qed

**Example 4.1.5.** The series $\sum_{n \geq 1} (-1)^n$ diverges.

**Proof.** The values of $s_n$ alternate between the values -1 and 0. \qed

**Example 4.1.6.** (Geometric Series) For $a \neq 0$, the geometric series $\sum_{n \geq 0} ar^n$ converges to $a/(1 - r)$ for $|r| < 1$ and diverges if $|r| \geq 1$.

**Proof.** $s_n = \frac{a - ar^n}{1 - r} \to \frac{a}{1 - r}$ for $n \to \infty$ if and only if $|r| < 1$. \qed

**Example 4.1.7.** (Telescoping Series) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2 + n} = 1$.

**Proof.** We look at the generic term and decompose it in partial fractions:

$$a_n = \frac{1}{n(n + 1)} = \frac{1}{n} - \frac{1}{n + 1}.$$ 

Therefore, $s_n = 1 - 1/(n + 1) \to 1$ for $n \to \infty$. \qed

In analogy to properties shown to hold for sequences, we have the following basic property of linearity for infinite series.
Theorem 4.1.8. If \( \sum_{n \geq 1} a_n \) converges to \( a \) and \( \sum_{n \geq 1} b_n \) converges to \( b \), then
\[
\sum_{n \geq 1} (c a_n + d b_n) \text{ converges to } ca + db \text{ for any constants } c, d.
\]

Exercises

Exercise 4.1.9. Restate Definition 4.1.2 with quantifiers. Negate it using quantifiers.


Exercise 4.1.11. Show that if \( \sum_{n \geq 1} a_n \) converges to \( a \) and \( \sum_{n \geq 1} b_n \) diverges, then \( \sum_{n \geq 1} (a_n + db_n) \) diverges for all constants \( d \neq 0 \).

Exercise 4.1.12. (a) Find two divergent series \( \sum_{n \geq 1} a_n \) and \( \sum_{n \geq 1} b_n \) such that \( \sum_{n \geq 1} (a_n + b_n) \) is convergent.

(b) Show that if the series \( \sum_{n \geq 1} (a_n + b_n) \) and \( \sum_{n \geq 1} (a_n - b_n) \) are both convergent, then so are the series \( \sum_{n \geq 1} a_n \) and \( \sum_{n \geq 1} b_n \).

Exercise 4.1.13. Show that \( \sum_{n \geq 1} a_n \) is convergent if and only if \( \sum_{n \geq n_0} a_n \) is convergent, where \( n_0 \) is a fixed positive integer \( n_0 \geq 1 \). Compare the sums of the two series.

Exercise 4.1.14. Show that if \( \sum_{n \geq 1} a_n \) converges to \( a \) and \( \sum_{n \geq 1} b_n \) converges to \( b \), then \( \sum_{n \geq 1} (a_n b_n) \) does not necessarily converge to \( ab \).

Exercise 4.1.15. Determine whether the series converge or diverge. In case of convergence, find the sum.

(1) \[ \sum_{n \geq 1} \frac{7}{2007^n}; \]

(2) \[ 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \ldots; \]

(3) \[ \sum_{n \geq 1} \frac{4^n + 1}{20^n}; \]

(4) \[ \sum_{k \geq 1} \frac{2}{4k^2 - 1}; \]
EXERCISE 4.1.16. Compute \( \lim_{n \to \infty} \sqrt[n]{2 \sqrt[2]{2 \cdots \sqrt[2]{2}}} \).

EXERCISE 4.1.17. The sequence \((x_n)\) is defined recursively by

\[
x_1 = 1, \quad x_{n+1} = \sum_{k=1}^{n} x_k.
\]

(a) Find the general formula of \(x_n\).
(b) Show that \( \lim_{n \to \infty} x_n = \infty \).

EXERCISE 4.1.18. Let \(a_n = 1/(n^2 + n + 1)\).
(a) Show that \(\arctan a_n = \arctan(n+1) - \arctan(n)\).
(b) Find \(\sum_{n=1}^{\infty} \arctan a_n\).

EXERCISE 4.1.19. Assume that the series \(\sum a_n\) is convergent to \(a\). Show that
(a) \(\sum (a_n + 2a_n)\) also converges and find its sum.
(b) \(\sum (a_n + 1)\) diverges to infinity.

4.2. A simple divergence test

THEOREM 4.2.1. If \(\sum a_n\) converges, then \(\lim_{n \to \infty} a_n = 0\). Equivalently, if \(\lim_{n \to \infty} a_n \neq 0\), then \(\sum_{n \geq 1} a_n\) diverges.

PROOF. The series being convergent simply means that \(s_n \to s\) for some \(s \in \mathbb{R}\), where \(s_n\) stand for the \(n\)th partial sums of the series. But, then \(s_{n-1} \to s\) as well and

\[a_n = s_n - s_{n-1} \to s - s = 0, \quad \text{as} \ n \to \infty.\]

Clearly, the second statement, which provides a simple sufficient condition for the divergence of a series, is equivalent to the first statement that we just proved.

REMARK 4.2.2. As we will see in the next section, having the limit of the general term \(a_n\) in the series equal to zero does not imply the convergence of the series \(\sum a_n\). An example of such series is

\[
\sum_{n \geq 1} \frac{1}{n}.
\]
Exercises

Exercise 4.2.3. Use Theorem 4.2.1 to show that \( \sum (-1)^n \) diverges.

Exercise 4.2.4. Use Theorem 4.2.1 to show that \( \sum \frac{n^2}{n^2 + 1} \) diverges.

Exercise 4.2.5. Can we use Theorem 4.2.1 to determine whether \( \sum \frac{1}{n} \) diverges?

Exercise 4.2.6. Assume that \( a_n \neq 0 \) for all \( n \in \mathbb{N} \) and \( \sum a_n \) converges. Investigate the convergence/divergence of the following series. In case of convergence, find the sum.

1. \( \sum \frac{1}{|a_n|} \);
2. \( \sum \cos a_n \);
3. \( \sum (a_n - a_{n+1}) \).

Exercise 4.2.7. The series \( \sum a_n \) has the partial sum \( s_n = \frac{n}{n + 2} \) for \( n \geq 1 \).

(a) What is the value of \( a_1 \)?
(b) Find the expression of \( a_n, n \geq 2 \).
(c) What is the value of the sum of the series \( \sum_{n=1}^{\infty} a_n \)?

4.3. The integral test and the comparison tests

In many situations we need to decide the convergence or divergence of a series indirectly. The tests listed below help us in this task. In particular, these tests will not provide directly the exact value of the sum of a series.

Theorem 4.3.1. (Integral Test) Let \( a_n = f(n) \), where \( f \) is a continuous, positive and non-increasing function for all \( x \geq n_0 \), with \( n_0 \) some fixed positive integer. Then the series \( \sum_{n \geq n_0} a_n \) and the improper integral \( \int_{n_0}^{\infty} f(x) \, dx \) have the same behavior, that is, they either both converge or both diverge.

Proof. Without loss of generality, assume that \( n_0 = 1 \). Comparing the area under the graph of \( f \) and above the \( x \)-axis on the interval \([1, n]\) with left and right-hand Riemann sums associated to sub-intervals of length one, gives

\[
0 \leq s_n - a_1 \leq \int_{1}^{n} f(x) \, dx \leq s_{n-1}.
\]

Here, \( s_n \) denotes the \( n \)th partial sum \( \sum_{k=1}^{n} a_k \). The conclusion follows by passing to the limit as \( n \to \infty \). \( \square \)
The same idea can be used to obtain error estimates when utilizing the integral test.

**Theorem 4.3.2.** (Error Estimates) Under the same conditions as in Theorem 4.3.1, if \( \sum_{n=1}^{\infty} a_n = s \) and \( r_n = s - s_n \) denotes the remainder of order \( n \), then
\[
\int_{n+1}^{\infty} f(x) \, dx \leq r_n \leq \int_{n}^{\infty} f(x) \, dx.
\]

**Example 4.3.3.** (\( p \)-series) Show that the series \( \sum_{n \geq 1} \frac{1}{n^p} \) converges for \( p > 1 \) and diverges for \( p \leq 1 \).

**Proof.** Let \( p > 1 \). Clearly \( a_n = f(n) \), where \( f(x) = \frac{1}{x^p} \) is positive and continuous for \( x \geq 1 \). Furthermore, \( f \) is decreasing for \( x \geq 1 \) since \( f'(x) = -px^{-p-1} < 0 \). Therefore, we can apply the integral test. We simply compute the improper integral
\[
\int_{1}^{\infty} \frac{1}{x^p} \, dx = \frac{1}{p-1} < \infty,
\]
and conclude that the series converges in this case.

If \( p = 1 \), we repeat the argument and get
\[
\int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{B \to \infty} \ln B = \infty,
\]
so in this case the series diverges.

If \( 0 < p < 1 \), we get
\[
\int_{1}^{\infty} \frac{1}{x^p} \, dx = \lim_{B \to \infty} \left( B^{1-p} \frac{B^{1-p} - 1}{1 - p} \right) = \infty,
\]
so again the series diverges.

Finally, if \( p \leq 0 \), then \( 1/n^p \geq 1 \) and we can apply the simplest divergence test.

Note that we have not computed the exact value of the sum of any \( p \)-series. Indeed, the finite answer for the corresponding improper integral does not represent the value of the sum. To compute the exact values of such sums, a more subtle approach is required. \(^1\)

**Example 4.3.4.** The series \( \sum_{n \geq 1} \frac{\ln n}{n} \) diverges.

\(^1\)A. Bényi, Finding the sums of harmonic series of even order, College Math Journal 36 (2005), 44-48
PROOF. Start by observing that $f(x) = \ln x / x$ is positive, continuous and decreasing for $x \geq 3$. The fact that the function $f$ is decreasing follows from $f'(x) = (1 - \ln x) / x^2 \leq 0$ for $x \geq e$. Therefore, we can use the integral test. We compute

$$\int_3^\infty \frac{\ln x}{x} \, dx = \lim_{B \to \infty} \left( \frac{(\ln B)^2}{2} - \frac{(\ln 3)^2}{2} \right) = \infty.$$

This proves that the integral, and hence the series, diverges. When computing the integral we used the substitution $u = \ln x$.

EXAMPLE 4.3.5. Determine the number of terms needed to approximate the sum $\sum_{n=1}^\infty \frac{1}{n^2}$ correct to within 0.1.

PROOF. We use Theorem 4.3.2. We know that the remainder $r_n = s - s_n$ of order $n$ satisfies

$$0 \leq r_n \leq \int_n^\infty \frac{1}{x^2} \, dx = \frac{1}{n}.$$

We want $1/n \leq 0.1$ or $n \geq 10$. Therefore, by taking $n \geq 10$ we achieve the required accuracy, that is

$$s \approx \sum_{n=1}^{10} \frac{1}{n^2} = 1.54976773117$$

is correct within 0.1.

**Theorem 4.3.6.** (Direct Comparison Test) Assume $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

(a) If $\sum_{n=1}^\infty a_n$ diverges, then $\sum_{n=1}^\infty b_n$ diverges.

(b) If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges.

The proof of this comparison test is left to you as an exercise; see Exercise 4.3.16. In your argument, you will have to make use of Theorem 3.4.14. Note also that, due to Exercise 4.1.13, Theorem 4.3.6 is equivalent to the following result.

**Theorem 4.3.7.** (Direct Comparison Test Revisited) Let $n_0 \in \mathbb{N}$ be fixed. Assume $0 \leq a_n \leq b_n$ for all $n \geq n_0$.

(a) If $\sum_{n=n_0}^\infty a_n$ diverges, then $\sum_{n=n_0}^\infty b_n$ diverges.

(b) If $\sum_{n=n_0}^\infty b_n$ converges, then $\sum_{n=n_0}^\infty a_n$ converges.

The most straightforward uses of the Comparison Test I are to show convergence or (divergence) of a series by comparing it to either a geometric series or an appropriate $p$-series.
Example 4.3.8. Investigate the convergence of the series \( \sum_{n=1}^{\infty} \frac{1}{2n^2 + 8n + 1} \).

Proof. We use Theorem 4.3.6. Note that \( 0 \leq \frac{1}{(2n^2 + 8n + 1)} \leq \frac{1}{(2n^2)} \) for all \( n \in \mathbb{N} \). By using that the 2-series converges, we conclude that this series is convergent as well.

Sometimes it is easier to use a variation of Theorem 4.3.6; the content of the next theorem is that the comparison can be reduced to that of an associated series which captures its “essential part”.

Theorem 4.3.9. (Limit Comparison Test) Assume \( a_n, b_n > 0 \) for all \( n \in \mathbb{N} \).

(a) Further assume that \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \) exists.

(a1) If \( L > 0 \), then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) have the same behavior.

(a2) If \( L = 0 \) and \( \sum_{n=1}^{\infty} b_n \) converges, then so is \( \sum_{n=1}^{\infty} a_n \).

(b) If \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \) and \( \sum_{n=1}^{\infty} b_n \) diverges, then so is \( \sum_{n=1}^{\infty} a_n \).

Proof. We will show that Comparison Test II is simply a consequence of Comparison Test I.

(a) Let us assume that \( \sum_{n=1}^{\infty} b_n \) converges to \( B \geq 0 \). In particular, this implies that for all \( n_0 \in \mathbb{N} \) we have

\[
\sum_{n=n_0}^{\infty} b_n \leq B.
\]

By hypothesis, we know that there exists \( N \in \mathbb{N} \) (fixed) such that for all \( n \in \mathbb{N} \) with \( n > N \) we have \( |a_n/b_n - L| < 1 \); here, we used the definition of limit for the sequence \( (a_n/b_n) \) with \( \epsilon = 1 \). In particular, we obtain that for all \( n \geq N + 1 \) we have

\[
\frac{a_n}{b_n} < L + 1 \Leftrightarrow a_n < (L + 1)b_n.
\]

Thus, with \( n_0 = N + 1 \), Theorem 4.3.7 implies now the convergence of \( \sum_{n \geq N+1} a_n \), which in turn is equivalent (again by Exercise 4.1.13) to the convergence of \( \sum_{n=1}^{\infty} a_n \). This proves one implication in (a1) and (a2) at once.

For the reverse implication in (a1), assume now that \( L > 0 \) and \( \sum_{n=1}^{\infty} a_n \) is convergent. Using the definition of limit for \( a_n/b_n \), this time with \( \epsilon = L/2 \), we obtain that for some \( N' \in \mathbb{N} \) and for all \( n > N' \) we have

\[
\left| \frac{a_n}{b_n} - L \right| < \frac{L}{2} \Rightarrow \frac{a_n}{b_n} > \frac{L}{2} \Rightarrow b_n < \frac{2}{L} a_n.
\]
From this point on, we repeat the argument above and conclude that the series \( \sum_{n \geq N'} b_n \) converges, thus the series \( \sum_{n \geq 1} b_n \) converges.

Part (b) follows from similar considerations. \( \square \)

**Example 4.3.10.** Investigate the convergence of the series \( \sum_{n=1}^{\infty} \frac{1}{2n^2 - \sqrt{8n}} \).

**Proof.** We use Theorem 4.3.9. Note that \( 0 \leq \frac{1}{(2n^2 - \sqrt{8n})} \) for \( n \geq 2 \) and that
\[
\lim_{n \to \infty} \frac{1/(2n^2 - \sqrt{8n})}{1/n^2} = \frac{1}{2} > 0.
\]
By using that the 2-series converges, we conclude that this series is convergent as well. \( \square \)

**Question 4.3.11.** Can we use Theorem 4.3.6 to arrive to the same conclusion?

**Example 4.3.12.** Show the divergence of the series in Example 4.3.4 using one of the comparison tests.

**Proof.** Simply observe that
\[
\lim_{n \to \infty} \frac{\ln n/n}{1/n} = \infty
\]
and that the 1-series diverges. Now apply part (b) of Theorem 4.3.9. \( \square \)

**Exercises**

You have learned several tests now, so you will have to determine the right test to use. There might be the case that, in some of the exercises, more than one test can be used.

**Exercise 4.3.13.** Investigate the convergence/divergence of the following series.

1. \( \sum_{n \geq 2} \frac{1}{n \ln n} \);
2. \( \sum_{n \geq 2} \frac{1}{n(\ln n)^2} \);
3. \( \sum_{n \geq 1} \frac{1}{n^2 + 1} \);
4. \( \sum_{n \geq 1} \frac{3n^2 + 17n - 5}{n^5 + 48n^4 + 2007} \);
5. \( \sum_{n \geq 1} \frac{2^n}{3^n + 4} \);
(6) \[ \sum_{n \geq 2} \frac{\ln n}{n^2}; \]

(7) \[ \sum_{n \geq 1} \frac{n^2}{e^n}; \]

(8) \[ \sum_{n \geq 1} \frac{2 + \cos n}{n}; \]

(9) \[ \sum_{n \geq 1} \frac{1}{\sqrt{n^2 + 1}}; \]

(10) \[ \sum_{n \geq 1} \frac{1}{\sqrt{n(n+1)(n+2)}}. \]

**Exercise 4.3.14.** Consider the series \( \sum_{n \geq 0} \frac{1 + \sin n}{10^n} \).

(a) Are all conditions in the integral test verified?

(b) Assuming that all conditions in the integral test are verified, explain in your own words how would you apply the integral test to the given series.

(c) Determine whether the series converges or diverges without using the integral test.

**Exercise 4.3.15.** Determine the number of terms needed to approximate the sum \( s \) of the 3-series correct to within \( 10^{-5} \). Find the first four digits after the decimal point of \( s \).

**Exercise 4.3.16.** Prove Theorem 4.3.6.

**Exercise 4.3.17.** Assume that \( \sum a_n \) is a convergent series and \( a_n > 0 \) for all \( n \in \mathbb{N} \). Decide whether the following series converge, diverge, or there is not enough information provided.

1. \[ \sum a_n^p, \text{ where } p > 1; \]
2. \[ \sum na_n; \]
3. \[ \sum \frac{a_n}{n}; \]
4. \[ \sum \sin(a_n). \]

**4.4. The ratio and root tests**

Recall that a geometric series \( \sum_{n \geq 0} ar^n \), with \( a \neq 0 \), is convergent if and only if \( |r| < 1 \). A special feature of such a series is that the ratio of any two consecutive terms stays constant and equal to the ratio \( r \). It turns out that many series, although not exactly geometric, behave similarly to such a series.

**Theorem 4.4.1.** (The Ratio Test) Let \( \sum a_n \) be a series of positive terms and assume that \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = R \] exists.
(a) If $R < 1$, then $\sum_{n \geq 1} a_n$ converges.

(b) If $R > 1$, then $\sum_{n \geq 1} a_n$ diverges.

(c) If $R = 1$, then the test is inconclusive.

Proof. Let us start by observing that although we cannot compare our series with a geometric series with ratio $R$ (the ratios $a_{n+1}/a_n$ could be a little more or little less than $R$), we can still use the Direct Comparison Test with a geometric series having an appropriate ratio $r$.

(a) Assuming $R < 1$, we have $\epsilon = (1 - R)/2 > 0$. By definition of limit, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\left| \frac{a_{n+1}}{a_n} - R \right| < \frac{1 - R}{2}.$$ 

In particular, for all $n > N$ we have

$$\frac{a_{n+1}}{a_n} < R + \frac{1 - R}{2} = \frac{1 + R}{2}.$$ 

Denote $r = (1 + R)/2$ and notice that $r \in (0, 1)$ (since $R \in [0, 1]$). Moreover, for all $n > N$, we have $a_{n+1} < ra_n$. Letting $n = N + 1$, then $n = N + 2$ and so on, we see that for all $n \geq N + 1$ we have

$$a_n \leq b_n = a_{N+1}r^{n-N-1};$$

that is, we have a direct comparison between the series $\sum_{n \geq N+1} a_n$ and the convergent geometric series $\sum_{k \geq 0} a_{N+1}r^k$. Thus Theorem 4.3.7 implies the convergence of $\sum_{n \geq N+1} a_n$ and, consequently the convergence of $\sum_{n \geq 1} a_n$; we used yet again Exercise 4.1.13 here.

(b) Assuming that $R > 1$, we proceed as above to find $N' \in \mathbb{N}$ such that for all $n > N'$ we have

$$\left| \frac{a_{n+1}}{a_n} - R \right| < \frac{R - 1}{2}.$$ 

In particular, for all $n > N'$ we have now

$$\frac{a_{n+1}}{a_n} > R - \frac{R - 1}{2} = \frac{1 + R}{2}.$$ 

Denoting again $r = (1 + R)/2$, we see that $r > 1$ and the geometric series $\sum_{k \geq 0} a_{N'+1}r^k$ diverges. By Theorem 4.3.7 we obtain that $\sum_{n \geq 1} a_n$ diverges as well.

(c) This is addressed in the next example. 

Example 4.4.2. Consider the 1-series and the 2-series. Why is it that we fall under Theorem 4.4.1, part (c)?
**4.4. THE RATIO AND ROOT TESTS**

**Proof.** In the case of the 1-series,

\[
\frac{a_{n+1}}{a_n} = \frac{n + 1}{n} \to 1,
\]

but by the integral test we know it is in fact divergent.

In the case of the 2-series, the limit of the ratio is again 1, yet by the integral test again, we know it is convergent. \(\square\)

**Example 4.4.3.** Investigate the convergence of the series \(\sum_{n=1}^{\infty} \frac{n!}{e^n}\).

**Proof.** Let \(a_n = n!/e^n\). We have

\[
\frac{a_{n+1}}{a_n} = \frac{n + 1}{n} \to \infty.
\]

Therefore, by using the Ratio Test, part (b), the series diverges. \(\square\)

**Example 4.4.4.** (geometric series revisited) The series \(\sum_{n=1}^{\infty} \frac{2^n}{3^n}\) converges.

**Proof.** Let \(a_n = 2^n/3^n\). We have

\[
\frac{a_{n+1}}{a_n} = \frac{2}{3} \to \frac{2}{3} < 1.
\]

Use now the Ratio Test, part (a). \(\square\)

In certain situations, computing the limit of the ratio of two consecutive terms in a series might turn out to be a delicate task. The following test offers an alternative to this computation.

**Theorem 4.4.5.** (The Root Test) Let \(\sum a_n\) be a series of positive terms and assume that \(\lim_{n \to \infty} \sqrt[n]{a_n} = R\) exists.

(a) If \(R < 1\), then \(\sum_{n \geq 1} a_n\) converges.

(b) If \(R > 1\), then \(\sum_{n \geq 1} a_n\) diverges.

(c) If \(R = 1\), then the test is inconclusive.

The proof of the Root Test, which is similar in spirit to the proof of the Ratio Test, is left to you as an exercise; see Exercise 4.4.15.

**Example 4.4.6.** Investigate the convergence of the series \(\sum_{n \geq 1} \frac{n}{e^n}\).
Proof. Note that both tests in this section can be applied.

Ratio Test: \( a_{n+1}/a_n = (n + 1)/ne \to 1/e < 1 \); the series converges by Theorem 4.4.1.

Root Test: \( \sqrt[n]{a_n} = \sqrt[n]{n}/e \to 1/e < 1 \); the series converges by Theorem 4.4.5.

\[ \Box \]

**Example 4.4.7.** The series \( \sum_{n \geq 1} \left( 2 + \frac{1}{n} \right)^n \) diverges.

**Proof.** In this case, it is much easier to use the Root Test since

\[
\sqrt[n]{\left( 2 + \frac{1}{n} \right)^n} = 2 + \frac{1}{n} \to 2 > 1.
\]

\[ \Box \]

**Question 4.4.8.** In Example 4.4.7, can you arrive to the same conclusion using the Ratio Test?

**Example 4.4.9.** Investigate the convergence of the series \( \sum_{n \geq 1} \frac{n^n}{e^{n^2}} \).

**Proof.** We use the Root Test. Our choice is determined by the occurrence of the \( n \)th powers. We have

\[
\sqrt[n]{\frac{n^n}{e^{n^2}}} = \frac{n}{e^n} \to 0 \text{ (by l'Hôpital's rule)}.
\]

Therefore, the series converges.

\[ \Box \]

**Exercises**

**Exercise 4.4.10.** Investigate the convergence of each series.

1. \( \sum_{n \geq 1} \frac{n^n}{n!} \);
2. \( \sum_{n \geq 1} \frac{n^n}{e^{2n}} \);
3. \( \sum_{n \geq 1} \frac{(n!)^2}{(2n)!} \);
4. \( \sum_{n \geq 1} \frac{n^{2007}}{2007^n} \);
5. \( \sum_{n \geq 1} \frac{n!}{(n + 2)!} \);
6. \( \frac{2\sqrt{2}}{9801} \sum_{n \geq 0} \frac{(4n)!(1103 + 26390n)}{(n!)^4(396)^{4n}} \).
Exercise 4.4.11. Let \( a \in \mathbb{R} \) and \( b \in \mathbb{N} \). Consider the series \( \sum_{n \geq 1} \frac{(n!)^a}{(bn)!} \).

Investigate the relationship between the parameters \( a, b \) that guarantees the convergence of this series.

Exercise 4.4.12. The sequence \( (a_n) \) is defined as follows: \( a_n = n^2/e^n \) for \( n \) odd and \( a_n = 2007/e^n \) for \( n \) even. Show that \( \sum_{n \geq 1} a_n \) is convergent.

Exercise 4.4.13. The terms of the series \( \sum a_n \) are defined recursively by \( a_1 = 2, a_{n+1} = \frac{5n+1}{4n+3} a_n \). Without finding explicitly \( a_n \), determine whether the series \( \sum_{n \geq 1} a_n \) converges or diverges.

Exercise 4.4.14. Same question as in the previous exercise, except that the recursive relation is \( a_1 = 1, a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n \).

Exercise 4.4.15. Prove Theorem 4.4.5.

Exercise 4.4.16. Can we use any of the tests in this section to decide the convergence of \( \sum_{n \geq 1} \frac{e^{1/n}}{n^2} \)? If not, what would be a good test to use?

4.5. Alternating series. Absolute and conditional convergence

The previous sections have been devoted almost in exclusivity to series with positive terms. We will now consider series that do not necessarily have this property anymore. As we will see, the behavior of such series could be very different from (similar) series which contain only positive terms. We start by investigating alternating series, which are series such that the signs of the terms alternate from positive to negative.

Definition 4.5.1. An alternating series is a series of the form

\[ a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n \geq 1} (-1)^{n+1} a_n, \]

where \( a_n > 0 \) for all \( n \geq 1 \).

We made the convention that the first term of the series is positive, then the next one is negative and so on. Certainly, we just as well could have started with a negative term, then have the next one be positive and so on. In the latter case we would have written \( \sum (-1)^n a_n \) with \( a_n > 0 \). The two notations are the same with the exception of an extra minus sign.

Example 4.5.2. The following are alternating series:

1. \( 1 - 2 + 3 - 4 + 5 - 6 + \cdots = \sum_{n \geq 1} (-1)^{n+1} n; \)
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\( (2) \quad 1 - 1/2 + 1/3 - 1/4 + \cdots = \sum_{n \geq 1} (-1)^{n+1} \frac{1}{n}; \)

\( (3) \quad -1 + 1/2 - 1/4 + 1/8 - \cdots = \sum_{n \geq 1} \frac{(-1)^n}{2^{n-1}}. \)

The series (2) in Example 4.5.2 is referred to as the alternating harmonic series or alternating 1-series.

Theorem 4.5.3. (Alternating Series Test) If \( a_n \) is non-increasing and \( a_n \to 0 \) as \( n \to \infty \), then \( \sum (-1)^{n+1} a_n \) is convergent.

Proof. The subsequences of even and odd-indexed partial sums, \( (s_{2n}) \) and \( (s_{2n+1}) \) respectively, satisfy the following inequalities:

\[ s_{2n-2} \leq s_{2n} \leq s_{2n+1} \leq s_{2n-1}. \]

In particular, both subsequences are bounded and monotonic. Therefore, by Theorem 3.4.14, both these subsequences are convergent. Since \( s_{2n+1} = s_{2n} + a_{2n+1} \) and \( a_{2n+1} \to 0 \) as \( n \to \infty \) we conclude that the two subsequences of partial sums have the same limit, which in turn implies that \( (s_n) \) is convergent. By definition, the limit of \( s_n \) represents the sum of the series.

Note that Theorem 4.5.3 provides only a sufficient condition for the convergence of a series. In general this condition is not necessary (see Example 4.5.9 below). Furthermore, as for the integral test, we can estimate the error \( r_n = s - s_n \) as follows.

Theorem 4.5.4. (Error Estimates) Assume that the conditions of Theorem 4.5.3 hold, and let \( s_n \to s \), where \( s_n \) denotes the nth partial sum of the alternating series \( \sum (-1)^{n+1} a_n \). Then

\[ |r_n| \leq a_{n+1} \]

and \( r_n \) has the sign of the first unused term \( a_{n+1} \).

Example 4.5.5. Investigate the convergence of the series in Example 4.5.2.

Proof. We have:

(1) Divergent; simply because \((-1)^{n+1} n \not\to 0 \) as \( n \to \infty \) (see Theorem 4.2.1).

(2) Convergent; we can use Theorem 4.5.3 in which \( a_n = 1/n \) (decreasing and positive).

(3) Convergent; we can use Theorem 4.5.3 in which \( a_n = 1/2^{n-1} \) (decreasing and positive). Alternately, simply recognize it as a geometric series with ratio \(-1/2\).
Example 4.5.6. How many terms are needed to guarantee that the \( n \)th partial sum of the series in Example 4.5.2 (2) and (3) are within \( 10^{-1} \) of the actual sum?

**Proof.** We use Theorem 4.5.4.

(2) If \( s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \), then \( |s - s_n| \leq 1/(n + 1) \leq 10^{-1} \) implies that \( n \geq 9 \). In other words, \( s_9 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + 1/9 \) approximates \( s \) within \( 0.1 \).

(3) We need \( a_{n+1} = 1/2^n \leq 0.1 \) or \( n \geq 4 \). We can check that our work is correct. The sum of the first 4 terms, \( s_4 = -1 + 1/2 - 1/3 + 1/4 = -7/12 \). Of course, for this series we know how to find \( s \) exactly (since it is a geometric series with ratio \(-1/2\)), \( s = -1/(1 - (-1/2)) = -2/3 \).

The difference \( s_4 - s = 2/3 - 7/12 = 1/12 < 1/10 \).

**Definition 4.5.7.** (a) We say that a series \( \sum a_n \) is **absolutely convergent** if \( \sum |a_n| \) is convergent.

(b) We say that a series \( \sum a_n \) is **conditionally convergent** if \( \sum a_n \) is convergent and \( \sum |a_n| \) is divergent.

**Theorem 4.5.8. (Absolute Convergence)** Any absolutely convergent series is a convergent series.

**Proof.** We use the obvious fact \(-|a_n| \leq a_n \leq |a_n|\). If we know that \( \sum |a_n| \) is convergent, the same holds true for \( \sum 2|a_n| \). But

\[
0 \leq a_n + |a_n| \leq 2|a_n|
\]

allows us to use the Direct Comparison Test (Theorem 4.3.6) and conclude that \( \sum (a_n + |a_n|) \) is convergent. This in turn implies that \( \sum a_n \) is convergent because \( a_n = (a_n + |a_n|) - |a_n| \) and both series \( \sum (a_n + |a_n|) \) and \( \sum |a_n| \) both converge. For the last implication we used Theorem 4.1.8.

**Example 4.5.9.** The series \( \sum_{n \geq 1} \frac{\cos n}{n^2} \) is absolutely convergent.

**Proof.** We use the Direct Comparison Test. We have \( 0 \leq |(\cos n)/n^2| \leq 1/n^2 \) and since the 2-series \( \sum 1/n^2 \) converges, the series of absolute values \( \sum \left| \frac{\cos n}{n^2} \right| \) must also converge. Note that the series considered does not satisfy the conditions in Theorem 4.5.3.
Example 4.5.10. (a) The alternating harmonic series is conditionally convergent. Indeed, convergence follows as in Example 4.5.5. It is not absolutely convergent, since the sum of the absolute values represents the 1-series, which we know is divergent.

(b) The alternating series \(-1 + 1/2 - 1/4 + \ldots\) is absolutely convergent, since the sum of absolute values is a geometric series with ratio 1/2.

Exercises

Exercise 4.5.11. Investigate the convergence of the following alternating series.

1. \( \sum_{n \geq 1} \frac{(-1)^n}{n!} \);
2. \( \sum_{n \geq 1} \frac{(-1)^n n}{n + 1} \);
3. \( \sum_{n \geq 1} \frac{(-1)^n e^n}{e^{2n}} \);
4. \( \sum_{n \geq 1} \frac{(-1)^n \ln(e^n)}{\ln(e^{2n})} \).

Exercise 4.5.12. Investigate the absolute convergence of the series in the previous exercise.

Exercise 4.5.13. For each series, decide whether it converges absolutely, converges or diverges.

1. \( \sum_{n \geq 1} \frac{\sin(n\pi + \pi/2)}{\sqrt{n}} \);
2. \( \sum_{n \geq 1} (-1)^{n+1} \sqrt{n} \);
3. \( \sum_{n \geq 1} \frac{(-1)^{n+1} (n^3 - 2n + 7)}{n^5 + 5n^3} \);
4. \( \sum_{n \geq 1} \frac{(-1)^n n^2}{e^{2n}} \).

Exercise 4.5.14. Show that the sum of two absolutely convergent series is absolutely convergent.

Exercise 4.5.15. Find two conditionally convergent series such that
(a) the sum of the two series is conditionally convergent;
(b) the sum of the two series is absolutely convergent.

Exercise 4.5.16. Let \((x_n)\) be some sequence of real numbers.

(a) Find and simplify the nth partial sum of \( \sum_{n \geq 1} (-1)^n (x_n + x_{n+1}) \).
(b) Use (a) to find the exact value of \( \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n} + \frac{1}{n+1} \right) \).

Exercise 4.5.17. For \( n \in \mathbb{N} \), let \( a_n = (-1)^{n+1} \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x} \, dx \).

(a) Prove that \( a_n > 0 \) for all \( n \in \mathbb{N} \).
(b) By making the substitution \( u = x - \pi \), prove that \( a_n > a_{n+1} \) for all \( n \in \mathbb{N} \).
(c) Prove that \( a_n \to 0 \) as \( n \to \infty \).
(d) Use parts (a)-(c) to prove that the improper integral \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \) is convergent.
CHAPTER 5

Power Series and Taylor Series

5.1. Power series

Let us begin by recalling what we have learned about a geometric series: it converges if and only if the absolute value of its ratio is strictly less than 1. Furthermore, we have a formula that computes the sum of such a geometric series. For example, for any $|x| < 1$, we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$ 

Similarly, if $c$ is some given real number, then for any $|x-c| < 1$ we have

$$\sum_{n=0}^{\infty} (x-c)^n = \frac{1}{1-(x-c)}.$$ 

Therefore, by summing infinitely many functions, namely positive integer powers of $x$, we get another real valued function defined on some interval in $\mathbb{R}$. The two series considered above are just examples of so-called power series. Such objects play an important role in approximating the values of complicated functions, or computing their derivatives and integrals. In the next section we will see explicit applications of the power series.

**Definition 5.1.1.** A power series about $c$ (or centered at $c$) is a series of the form

$$\sum_{n \geq 0} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots$$

The sequence $(b_n)$ of real numbers is referred to as the sequence of coefficients (or simply, the coefficients) of the power series.

The basic question that we will concern ourselves with for power series is the following: for what values of $x$ does the series converge? This question can be rephrased into determining the domain of the real valued function representing the sum of the given power series.

**Example 5.1.2.** The series $\sum_{n \geq 0} x^n$ is a power series about $c = 0$. Its coefficients are $b_n = 1$ for all $n \geq 0$. The power series converges for all $|x| < 1$. Furthermore, we know that the sum of the series is $f(x) = 1/(1-x)$. 
This is equivalent to saying that the domain of \( f \) (which represents the sum of the power series above) is restricted to the open interval \((-1, 1)\).

**Example 5.1.3.** The power series \( \sum_{n \geq 0} \frac{1}{2^n} (x - 1)^n \) is centered at \( c = 1 \). Its coefficients are \( b_n = 1/2^n \). It is in fact a geometric series with ratio \( r = (x - 1)/2 \). Thus, the series converges as long as \( \left| \frac{x - 1}{2} \right| < 1 \Leftrightarrow |x - 1| < 2 \Leftrightarrow -1 < x < 3 \).

The sum of the series is the function

\[
    f(x) = \frac{1}{1 - \frac{x - 1}{2}} = \frac{2}{3 - x}
\]

which is defined on \((-1, 3)\).

The previous examples are representative in the sense that they provide a pretty good intuition of how to proceed in general.

**Theorem 5.1.4.** Let \( \sum_{n \geq 0} b_n (x - c)^n \) be a power series and assume that

\[
    \lim_{n \to \infty} \left| \frac{b_n}{b_{n+1}} \right| = R
\]

exists (or it is possibly \( \infty \)). Then the series converges absolutely for \( |x - c| < R \) and diverges for \( |x - c| > R \). If \( R = \infty \), the series converges absolutely for all \( x \in \mathbb{R} \).

**Proof.** It is trivial that the series converges absolutely for \( x = c \). Let then \( x \neq c \). The result is now a simple consequence of the Ratio Test applied to the series \( \sum a_n \), where \( a_n = |b_n (x - c)^n| \). Indeed, notice that

\[
    \frac{a_{n+1}}{a_n} = \left| \frac{b_{n+1}}{b_n} \right| \frac{|x - c|}{R} \to \frac{|x - c|}{R}.
\]

Thus the series \( \sum a_n = \sum |b_n (x - c)^n| \) converges when \( |x - c|/R < 1 \) and diverges when \( |x - c|/R > 1 \). It is worthwhile noting that by virtue of the Ratio Test we cannot conclude anything at the “end-points” where \( |x - c| = R \), that is for \( x \in \{c - R, c + R\} \). \( \square \)

The following result tells us exactly what are the situations that we can encounter when dealing with a power series. We note that the \( R \) in Theorem 5.1.5 is not necessarily computed as in Theorem 5.1.4. However, in most practical situations that is precisely how we will find the \( R \).

**Theorem 5.1.5.** Given a power series \( \sum_{n \geq 0} b_n (x - c)^n \), there are three distinct cases that can occur.

(a) There exists a number \( R > 0 \), called the radius of convergence, such that the series converges absolutely for \( |x - c| < R \) and diverges for \( |x - c| > R \). The interval \((c - R, c + R)\), and possibly its end-points (if the series converges there) form the so-called interval of convergence. The end-points
of the interval \((c - R, c + R)\) need to be tested separately for convergence, typically by substituting them back in the power series.

(b) The series converges absolutely for every \(x \in \mathbb{R}\). In this case \(R = \infty\) and the interval of convergence is \(\mathbb{R}\).

(c) The series converges at \(x = c\) and diverges elsewhere. In this case \(R = 0\) and the interval of convergence is the singleton \(\{c\}\).

As we pointed out already above, the main issue here is finding the radius of convergence. This will be done typically by repeating the argument of Theorem 5.1.4 that used the Ratio Test; equivalently, because of the appearance of the \(n\)th powers, one can think about the Root Test. Then, once we know that \(R\) is finite and non-zero, we will have to test separately for convergence or divergence at the end-points of the interval of convergence. To this end, we will use one of the other tests learned for series of real numbers, such as the comparison, integral or alternating series test. Let us look at some specific examples.

**Example 5.1.6.** Find the interval of convergence of the series

\[
\sum_{n \geq 0} \frac{(x - 1)^n}{n!}.
\]

**Proof.** We use the Ratio Rest. We have

\[
\frac{|x - 1|^n}{n+1} \rightarrow 0 < 1 \text{ as } n \rightarrow \infty
\]

for all real \(x\). Therefore the series converges absolutely for all \(x\), that is the interval of convergence is \(\mathbb{R}\), the set of all reals. The radius of convergence is \(R = \infty\).

**Example 5.1.7.** For what values of \(x\) does the power series

\[
\sum_{n \geq 0} n!(x - 1)^n
\]

converge?

**Proof.** From the Ratio Test, we get

\[
\frac{|x - 1|(x - 1)^{n+1}}{n!(x - 1)^n} = (n+1)|x - 1| \rightarrow \infty \text{ as } n \rightarrow \infty
\]

for all \(x \neq 1\). Therefore the series converges only at \(x = 1\). The radius of convergence is \(R = 0\).

**Example 5.1.8.** Find the interval of convergence of the power series

\[
\sum_{k \geq 1} \frac{x^k}{k^2 3^k}.
\]
Proof. We use yet again the Ratio Test:

\[
\left| \frac{x^{k+1}}{(k+1)3^{k+1}} \right| = \frac{|x|}{3} \frac{k^2}{(k+1)^2} \to \frac{|x|}{3} \text{ as } k \to \infty.
\]

Notice that the Root Test works equally well here:

\[
\sqrt[k]{\frac{x^k}{k^23^k}} = \frac{|x|}{3(\sqrt[k]{k})^2} \to \frac{|x|}{3} \text{ as } k \to \infty.
\]

We know that the series converges absolutely for \(|x|/3 < 1\) or \(|x| < 3\) or \(x \in (-3, 3)\). The radius of convergence is \(R = 3\). We need to test however the end-points as well.

If \(x = 3\), then the series becomes \(\sum_{k \geq 1} \frac{3^k}{k^23^k} = \sum_{k \geq 1} \frac{1}{k^2}\) which is a 2-series, hence convergent.

If \(x = -3\), then the series is \(\sum_{k \geq 1} \frac{(-3)^k}{k^23^k} = \sum_{k \geq 1} \frac{(-1)^k}{k^2}\) which is absolutely convergent because the sum of the absolute values of the terms in the series is again a 2-series.

Therefore, the interval of convergence is the closed interval \([-3, 3]\). \(\Box\)

Power series have the remarkable property that if they are convergent to some function \(f\) (defined on the interval of convergence) then \(f\) is differentiable on the interior of the interval of convergence. Furthermore, we can differentiate and integrate power series term-by-term. More precisely, we have the following statement.

**Theorem 5.1.9.** Let

\[ f(x) = \sum_{n=0}^{\infty} b_n(x-c)^n \]

be a function defined by the sum of a power series with positive radius of convergence \(R\). Then \(f\) is differentiable (hence continuous) at each point in the open interval \((c-R, c+R)\) and for all \(x \in (c-R, c+R)\) we have

\[ f'(x) = \sum_{n=0}^{\infty} b_n n(x-c)^{n-1} = \sum_{n=0}^{\infty} b_{n+1}(n+1)(x-c)^n \]

and

\[ \int_c^x f(t) \, dt = \sum_{n=0}^{\infty} b_n \frac{(x-c)^{n+1}}{n+1}. \]

The proof of Theorem 5.1.9 is non-trivial, and you should recognize that it does not follow from the usual rules that you have learned about differentiation and integration that state: the derivative (integral) of a sum of functions is the sum of the derivatives (integrals) of the functions. Indeed,
we deal here with infinite series, which are defined through a limiting process and it is not true in general that we can commute the operation of limit with that of taking a derivative or integral. Moreover, differentiation or integration term-by-term fails in general for series that are not power series (see Exercise 5.1.17).

Here are a couple of examples where term-by-term differentiation or integration can be useful.

**Example 5.1.10.** Consider the power series

\[ 1 - x + x^2 - x^3 + \cdots = \frac{1}{1 - (-x)}, \]

with convergence for all \( x \in (-1, 1) \). Apply now term-by-term integration to get

\[ \ln(1 + x) + C = \int \frac{1}{1 + x} \, dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \]

for \(-1 < x < 1\). In particular, for \( x = 0\), we get \( C = 0\). Therefore

\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots. \]

Note that the series converges at \( x = 1 \) as well (since it satisfies the alternating series test). This shows, in particular, that

\[ \ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}. \]

**Example 5.1.11.** Consider now the series

\[ 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2}. \]

This series converges for \( x^2 < 1 \) or \( x \in (-1, 1) \). Term-by-term integration gives

\[ \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots. \]

The power series converges also at \( x = \pm 1 \) because of the alternating series test. Equivalently, if we are asked to identify the function \( f(x) \) for which

\[ f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \]

we reverse our argument, use term-by-term differentiation and obtain

\[ f'(x) = 1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2}. \]

The function that has its derivative equal to \( 1/(1 + x^2) \) is then \( f(x) = \arctan x + C \), but since \( f(0) = 0 \) we conclude that \( C = 0 \), and \( f(x) = \arctan x \).
Exercises

Exercise 5.1.12. Determine the interval and radius of convergence of the power series below. When possible, use techniques similar to those in Examples 5.1.10 and 5.1.11 to find the sum of the power series.

(1) $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$;
(2) $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$;
(3) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{\ln(n+2)}$;
(4) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n (x + e)^n$;
(5) $\sum_{n=0}^{\infty} (1 + n)x^n$.

Exercise 5.1.13. Consider the power series $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot \cdots \cdot (2n)}{1 \cdot 3 \cdot \cdots \cdot (2n-1)} x^n$.

(a) Can $[0, 1)$ be the interval of convergence of this series?
(b) Find the interval of convergence of the series.
(c) If $f(x)$ denotes the sum of the series, express $\int_0^{1/2} f(x) \, dx$ as the sum of an infinite series of positive numbers.

Exercise 5.1.14. Show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real $x$.

Exercise 5.1.15. If $f(x)$ denotes the sum of the series in Exercise 5.1.14, find the power series that equals its derivative $f'(x)$. What is the relationship between $f(x)$ and $f'(x)$?

Exercise 5.1.16. Find the function $f(x)$ that represents the sum of the power series in Exercise 5.1.14. Use this fact to find the sum of the series

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

and

$$1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

Exercise 5.1.17. Consider the series $\sum_{n=1}^{\infty} \frac{\sin(n^4 x)}{n^3}$.

(a) Find its interval of convergence.
(b) Show that the series obtained through “term-by-term differentiation” diverges for all $x$. 

(c) Does part (b) contradict the “term-by-term differentiation” in Theorem 5.1.9?

EXERCISE 5.1.18. Assume that \( y = \sum_{n=0}^{\infty} a_n x^n \) solves the second order differential equation \( y'' - y' - y = 0 \). Find a recursive relation for the coefficients \( a_n \).

EXERCISE 5.1.19. The Bessel function of order 0 is defined as
\[
J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.
\]

(a) What is the domain of the function \( J_0 \)?
(b) Show that \( J_0 \) solves the linear differential equation \( xy'' + y' + xy = 0 \).

## 5.2. Taylor series

We will now address an issue previously encountered in single variable differential calculus: how do we approximate a function \( f(x) \) around a given point \( x = c \)? Recall that the tangent line at a point \((c, f(c))\) on the graph of a differentiable function provides the best linear approximation of \( f(x) \) for \( x \) very close to \( c \), that is
\[
x \approx c \Rightarrow f(x) \approx f(c) + f'(c)(x - c).
\]
Graphically, we see that, by zooming in more and more around the given point, the graph of \( f \) and that of the tangent line at the given point are indistinguishable.

Similarly, we might ask what are the best higher order polynomial approximations of \( f \) around a given value \( x = c \). The answer is provided by the so-called Taylor polynomials.

**Definition 5.2.1.** Let \( f \) be an infinitely differentiable function on some open interval that contains a given value \( x = c \). For all \( n \geq 1 \), the Taylor polynomial of order \( n \) associated to \( f \) at \( x = c \) is given by
\[
T_n(x; f, c) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k.
\]

Since in practice the function \( f \) and the “center” \( c \) are given, we prefer to use the simpler notation \( T_n(x) \) to denote this Taylor polynomial of order \( n \).

**Definition 5.2.2.** The Taylor series associated to \( f \) at \( x = c \) is the power series that has the Taylor polynomials \( T_n(x) \) as its partial sums, namely
\[
\sum_{k \geq 0} \frac{f^{(k)}(c)}{k!}(x - c)^k.
\]
When \( c = 0 \), the Taylor polynomials (or series) are also referred to as the Maclaurin polynomials (or series) associated to \( f \). As the values of \( n \) become larger and larger, the Taylor polynomials \( T_n(x) \) associated to \( f \) at \( c \) approximate better and better the function \( f(x) \) for \( x \) very close to \( c \). Note also that \( T_1(x) \) represents the linear (tangent line) approximation of \( f \) at \( c \).

A natural question arises. **What does the sum of the Taylor series associated to \( f \) equals to?** As we shall soon see, in many common situations we can assert that the Taylor series converges to the function that generates it. This is however not always the case, see Exercise 5.2.13.

**Example 5.2.3.** Find the Maclaurin series associated to \( f(x) = e^x \). Find the Maclaurin polynomials associated to \( f \).

**Proof.** Simply note that all the derivatives of \( f(x) \) are the same, \( f^{(k)}(x) = e^x \) for all \( k \geq 1 \). Therefore, the Taylor series associated to \( f \) at 0 (which is by definition the Maclaurin series) is

\[
\sum_{k=0}^{\infty} \frac{x^k}{k!}.
\]

Note also that the radius of convergence of this series is \( R = \infty \). The Maclaurin polynomial of order \( n \) is

\[
T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.
\]

\[\square\]

**Example 5.2.4.** Find the Maclaurin series associated to \( f(x) = \sin x \).

**Proof.** We need to compute the derivatives of all order at \( x = 0 \). The values of these derivatives depend on the parity of the order: \( f^{(2k)}(0) = 0 \), \( f^{(2k+1)}(0) = (-1)^k \). Therefore the Maclaurin series has only terms that are odd powers of \( x \):

\[
\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.
\]

Note again that the radius of convergence of this series is \( R = \infty \). \[\square\]

We now come to the crucial question we asked above: when do we expect a Taylor series \( T_n(x) \) to converge to the function \( f(x) \) that generated it?

**Theorem 5.2.5.** (Taylor’s formula) **Let** \( R > 0 \) **and assume** \( f \) **is infinitely differentiable on** \( (c - R, c + R) \). **Then, for all** \( x \in (c - R, c + R) \), **we have**

\[
f(x) = T_n(x) + R_n(x),
\]

where the remainder of order \( n \) is given by

\[
R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(x-t)^n \, dt.
\]
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In particular,

\[ |f(x) - T_n(x)| \leq M_n(f; c, x) \frac{|x - c|^{n+1}}{(n+1)!}, \]

where \( M_n(f; c, x) \) is the maximum of \( |f^{(n+1)}(t)| \) over the interval with endpoints \( c \) and \( x \).

**Proof.** Let us start by proving that the identity \( f(x) = T_n(x) + R_n(x) \) holds for \( n = 0 \). Indeed, by using the Fundamental Theorem of Calculus, we can write

\[ f(x) - T_0(x) = f(x) - f(c) = \int_c^x f'(t) \, dt = R_0(x). \]

To obtain the identity for \( n = 1 \), we will take the remainder \( R_0(x) \) and apply integration by parts to it:

\[
\begin{align*}
\int_c^x f'(t) \, dt &= \int_c^x (t - x)'f'(t) \, dt = (t - x)f'(t) \bigg|_c^x - \int_c^x (t - x)f''(t) \, dt \\
&= f'(c)(x - c) + \int_c^x (x - t)f''(t) \, dt.
\end{align*}
\]

Thus, tracking back, we see that we have obtained

\[ f(x) - f(c) = f'(c)(x - c) + \int_c^x (x - t)f''(t) \, dt \iff f(x) = T_1(x) + R_1(x). \]

Continue this process. By integrating by parts in the integral defining \( R_1(x) \) we will obtain the identity for \( n = 2 \), and so on. The general case is now proved using the method of mathematical induction.

Finally, the estimate on \( |f(x) - T_n(x)| = |R_n(x)| \) follows easily by noticing that

\[ |R_n(x)| \leq \frac{M_n(f; c, x)}{n!} \int_c^x (x - t)^n \, dt = M_n(f; c, x) \frac{|x - c|^{n+1}}{(n+1)!}. \]

Before continuing, let us point out that using the generic form of the mean value theorem from single variable calculus, \( R_n \) can also be expressed in its so-called Cauchy form of the remainder,

\[ R_n(x) = \frac{f^{(n+1)}(v)}{n!} (x - v)^n (x - c), \]

where \( v \) is some number between \( c \) and \( x \).

Returning to Taylor’s formula, we expect that a good control of the higher derivatives of \( f \) will imply a good control of the remainder \( R_n(x) \) for large \( n \). The following theorem confirms this intuition and also answers our main question.
THEOREM 5.2.6. (Taylor’s theorem) Assume that $f$ is an infinitely differentiable function on $(c-R, c+R)$ and there is a positive constant $M(c, x)$ such that $M_n(f; c, x) \leq M(x, c)$ for all $n$ sufficiently large and all $x \in (c-R, c+R)$. Then, the Taylor series of order $n$ associated to $f$ at $c$ converges to $f(x)$, that is
\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k.
\]

PROOF. Simply observe that the partial sum of order $n$ of the Taylor series is $T_n(x)$. Fix $x \in (c-R, c+R)$. By Theorem 5.2.5 and the hypothesis we conclude that
\[
|f(x) - T_n(x)| = |R_n(x)| \leq M(x, c) \frac{|x - c|^{n+1}}{(n+1)!} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
This implies $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, which is equivalent to saying that the Taylor series converges to $f(x)$.

The important thing to notice in the argument of Taylor’s Theorem 5.2.6 is that as long as we are able to control the remainder
\[
\lim_{n \rightarrow \infty} R_n(x) = 0,
\]
the Taylor polynomials will converge back to the function $f$ that generated them.

Let us now revisit the previous two examples and see why the Taylor series we have obtained converge to the functions that generated them in each of those situations.

EXAMPLE 5.2.7. The Maclaurin series associated to $f(x) = e^x$ converges to $e^x$ for all real values of $x$.

PROOF. Simply observe that for all $t$ in the interval with end-points $x$ and 0 ($c = 0$ for the Maclaurin series), we have for all $n \in \mathbb{N}$:
\[
|f^{(n+1)}(t)| = e^t \leq 1 + e^x = M(x, 0).
\]
Therefore, we can apply Theorem 5.2.6 to conclude that for all $x \in \mathbb{R}$ we have
\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.
\]
In particular, by letting $x = 1$ in Example 5.2.7 we obtain the following identity:
\[
(5.2.1) \quad e = \sum_{k=0}^{\infty} \frac{1}{k!}.
\]
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Example 5.2.8. The Maclaurin series associated to \( f(x) = \sin x \) converges to \( \sin x \) for all real values of \( x \).

Proof. Clearly, for all \( n \) and all \( t \) in the interval with end-points \( x \) and 0, we have \( |f^{(n+1)}(t)| \leq 1 = M \) and we can apply Theorem 5.2.6 again. We can therefore write for all \( x \in \mathbb{R} \):

\[
\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.
\]

These Taylor series expansions are extremely useful when trying to compute certain integrals or limits involving non-elementary functions. Let us exemplify how they can be used in specific situations.

Example 5.2.9. Compute \( \int e^x^3 \, dx \).

Proof. We use the fact that, for all \( x \in \mathbb{R} \), we have

\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.
\]

Therefore,

\[
e^x^3 = \sum_{k=0}^{\infty} \frac{x^{3k}}{k!},
\]

and using Theorem 5.1.9,

\[
\int e^x^3 \, dx = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{k!(3k + 1)} + C.
\]

Note that the methods learned before in integral calculus such as integration by parts, substitution, etc do not apply to this problem. Try using them to convince yourself!

Example 5.2.10. Compute

\[
\lim_{x \to 0} \frac{e^{x^3} - 1 - x^3 - x^6/2}{x^9}.
\]

Proof. We use the Taylor expansion previously found to write

\[
e^{x^3} - 1 - x^3 - \frac{x^6}{2} = \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots
\]

Therefore,

\[
\lim_{x \to 0} \frac{e^{x^3} - 1 - x^6/2}{x^9} = \lim_{x \to 0} \left( \frac{1}{3!} + \frac{x^3}{4!} + \cdots \right) = \frac{1}{6}.
\]
QUESTION 5.2.11. Can we use l'Hôpital's rule to compute the limit in the previous example? If yes, do so and then compare the two approaches. Is there an advantage of one approach over the other?

Exercises

EXERCISE 5.2.12. Find the Taylor series associated to \( f(x) \) at \( x = c \). Find the interval of convergence of each series and show that it converges to \( f(x) \) on its interval of convergence.

1. \( f(x) = 1 + x + x^2, \ c = -1 \);
2. \( f(x) = \cos x, \ c = 0 \);
3. \( f(x) = e^x, \ c = 1 \);
4. \( f(x) = x \cos x, \ c = 0 \);
5. \( f(x) = (x - 1)^2 e^x, \ c = 1 \);
6. \( f(x) = \cos(2x), \ c = 0 \);
7. \( f(x) = \cos^2(x), \ c = 0 \);
8. \( f(x) = \ln x, \ c = 1 \);
9. \( f(x) = \ln(1 + x), \ c = 0 \).

EXERCISE 5.2.13. Find the Maclaurin series associated to \( f(x) \) defined as \( f(x) = e^{-x^2}, x \neq 0 \) and \( f(0) = 0 \). Show that the Maclaurin series converges to \( f(x) \) only at \( x = 0 \).

EXERCISE 5.2.14. Express \( \int_x^0 \sin(t^2) \, dt \) as a power series around 0. Use this fact to express the definite integral \( \int_{-1}^0 \sin(t^2) \, dt \) as an infinite series of real numbers.

EXERCISE 5.2.15. By following the same approach as in the previous exercise, express \( \int_0^1 \frac{\sin x}{x} \, dx \) as an infinite series of real numbers.

EXERCISE 5.2.16. Use a known Taylor series to find the value of the limit.

1. \( \lim_{x \to 0} \frac{\cos x - 1}{x^2} \);
2. \( \lim_{x \to 0} \frac{\cos x - 1 - x^2/2}{x^4} \);
3. \( \lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} \);
4. \( \lim_{x \to 0} \frac{\sin x - x}{x^3} \);
5. \( \lim_{x \to 0} \frac{\arctan x - x}{x^3} \);
6. \( \lim_{x \to 1} \frac{\sin(x - 1) - x + 1}{(x - 1)^3} \).
Exercise 5.2.17. Let $R > 0$ and assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-R, R)$. Show that $n!a_n = f^{(n)}(0)$.

Exercise 5.2.18. Use the Maclaurin series associated to the functions $\sin x$, $\cos x$ and $e^{ix}$, where $i^2 = -1$, to prove Euler’s identity:

\[ e^{ix} = \cos x + i \sin x. \]

5.3. A proof of the irrationality of $e$

Many of you are familiar with the fact that $\sqrt{2}$ is irrational, meaning that we cannot express it as a ratio of two natural numbers; more precisely, for all $m, n \in \mathbb{N}$, $\sqrt{2} \neq \frac{m}{n}$. Two other famous irrational friends are $\pi$ and $e$. But how can one prove their irrationality?

In this section, we apply some of the knowledge about series in Chapter 4 and Sections 5.1 and 5.2 to show that the irrationality of $e$ follows in a rather straightforward way. We begin by pointing out that $e$ is not an integer. Indeed, from the identity (5.2.1), we have

\[ 2 = \frac{1}{0!} + \frac{1}{1!} < e = \sum_{k=0}^{\infty} \frac{1}{k!} \]

and, using telescoping series as in Example 4.1.7, we can also write

\[ e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots < 2 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots = 3. \]

Thus $2 < e < 3$, and since 2 and 3 are consecutive integers, we conclude that $e \not\in \mathbb{N}$.

Just as in the proof of the irrationality of $\sqrt{2}$, we proceed by way of contradiction and assume that $e$ is rational, that is,

\[ e = \frac{m}{n} \]

for some $m, n \in \mathbb{N}$ with $n > 1$ (note that $n = 1$ would mean $e$ is an integer, which we already proved is not the case). But then

\[ \alpha := n! \left( e - \sum_{k=0}^{n} \frac{1}{k!} \right) = (n-1)!m - \sum_{k=0}^{n} (k+1)(k+2) \cdots n \]

must be a integer. Now, since for each fixed $n \in \mathbb{N}$ we have

\[ e > \sum_{k=0}^{\infty} \frac{1}{k!} > \sum_{k=0}^{n} \frac{1}{k!}, \]

we clearly see that $\alpha > 0$.

Furthermore,

\[ \alpha = n! \left( \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} \right) = \sum_{k=n+1}^{\infty} \frac{n!}{k!}, \]
Let us take a closer look at this last series. Observe that each individual term in the series (with \( k \geq n + 1 \)) can be estimated from above as follows:

\[
\frac{n!}{k!} = \frac{1}{(n+1)(n+2) \cdots k} \leq \frac{1}{(n+1)(n+1) \cdots (n+1)} \text{,}
\]

that is

\[
\frac{n!}{k!} \leq \frac{1}{(n+1)^{k-n}}.
\]

The above inequality is in fact strict for all \( k \geq n + 2 \). Putting everything together, and changing the summation index via \( l = k - n \), we obtain

\[
\alpha \leq \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} = \sum_{l=1}^{\infty} (n+1)^{-l}.
\]

We are yet again in familiar territory: the very last series is a geometric series (see Example 4.1.6), which we know how to sum! Indeed,

\[
\sum_{l=1}^{\infty} (n+1)^{-l} = \frac{(n+1)^{-1}}{1 - (n+1)^{-1}} = \frac{1}{n}.
\]

We conclude that \( \alpha \leq 1/n < 1 \). Summarizing, \( \alpha \) defined above must be an integer such that \( 0 < \alpha < 1 \), which is a contradiction. This shows that our initial assumption about \( e \) being rational is false, hence \( e \) is irrational.