2.14.13. The Heine-Borel property

Any closed interval of \( \mathbb{R} \) has the Heine Borel property.

**Proof.** Let \( \mathcal{G} \) be an open cover of \([a, b], a < b\), i.e., \([a, b] \subset \bigcup \{G : G \in \mathcal{G}\}\) and each \( G \in \mathcal{G} \) is open. We want to show that there is a finite subfamily of \( \mathcal{G} \) that is also an open cover of \([a, b]\).

**Step 1.** If \( x \in [a, b) \), then there exists \( G \in \mathcal{G} \) and \( y \in [a, b), y > x \), such that \([x, y] \subset G\).

Indeed, since \([a, b] \subset \bigcup \{G : G \in \mathcal{G}\}\), and \( x \in [a, b) \), there exists \( G \in \mathcal{G} \) such that \( x \in G \). But \( x \neq b \) and \( G \) is open, thus giving us \([x, c) \subset G\) for some \( c \in [a, b) \). Pick any \( y \in (x, c) \) to obtain \([x, y] \subset G\).

**Step 2.** Let \( C = \{y \in (a, b) : \exists n \in \mathbb{N}, \exists G_1, ..., G_n, [a, y] \subset \bigcup_{j=1}^{n} G_j\} \).

By Step 1 applied to \( x = a \), we find \( y \in (a, b) \) and \( G \in \mathcal{G} \) such that \([x, y] \subset G\). Since \( C \) is bounded above, \( \sup C \) exists. Let \( c = \sup C \). Clearly, \( a < c \leq b \).

**Step 3.** \( c \in C \).

To show \( c \in C \) is equivalent to proving that \([a, c]\) can be covered by finitely many open sets in \( \mathcal{G} \). Since \( c \in (a, b] \), there exists \( G \in \mathcal{G} \) such that \( c \in G \). But \( G \) is open, so there exists \( d \in [a, b) \) such that \((d, c] \subset G\). If \( c \notin C \), then there must exist some \( z \in C \cap (d, c) \); otherwise, \( d \) would be a smaller upper bound of \( C \) than \( c \), which is impossible. But \( z \in C \) implies that there exist \( G_1, ..., G_n \in \mathcal{G} \) such that \([a, z] \subset \bigcup_{j=1}^{n} G_j \).

Moreover, \([z, c] \subset (d, c] \subset G\), so \([a, c] = [a, z] \cup [z, c] \subset \bigcup_{j=1}^{n+1} G_j \), where we let \( G_{n+1} = G \).

**Step 4.** \( c = b \).

Assume that \( c < b \). By Step 1, there exists \( y \in [a, b], y > c \), such that \([c, y] \subset G \) for some \( G \in \mathcal{G} \). But by Step 3, \( c \in C \), that is \([a, c] \) can be covered by a finite subfamily of \( \mathcal{G} \). This implies that \([a, y] = [a, c] \cup [c, y] \) can be also covered by a finite subfamily of open sets, so \( y \in C \). This however contradicts the fact that \( y > c = \sup C \).

Recalling again the definition of our set \( C \) from Step 2, we see that we proved that \([a, b] \) can be covered by a finite subfamily of \( \mathcal{G} \). The proof is complete.

\( \square \)