1. Consider the following operators on $l^2$:

\[ T_1((x_n)) = (n^{-1}(n-1)x_n), \]
\[ T_2((x_n)) = (i^n x_n), i^2 = -1, \]
\[ T_3((x_n)) = (x_2, x_3, ...). \]

(a) Prove that $T_j \in \mathcal{B}(l^2)$, $j = 1, 2, 3$.

(b) Find $T_j^*$, $j = 1, 2, 3$.

(c) Investigate whether the operators $T_j$, $j = 1, 2, 3$, are self-adjoint, unitary, or normal.

2. Let $H$ be a Hilbert space and $T \in \mathcal{B}(H)$. Show that the following conditions are equivalent: (1) $\|T\| \leq 1$; (2) $T^*T \leq I_H$; (3) $TT^* \leq I_H$.

3. Let $H$ be a Hilbert space, $M \subset H$ a closed linear subspace, and $P = P_M$ the projector onto $M$. Let $T \in \mathcal{B}(H)$. Prove the following.

(a) $M$ is invariant under $T$, i.e., $T(M) \subset M$, if and only if $PTP = TP$.

(b) $M$ is invariant under both $T$ and $T^*$ if and only if $T$ commutes with $P$, i.e., $TP = PT$. 