Project I: Uniformly convex spaces

Definition. Let $X$ be a Banach space. $X$ is called uniformly convex if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in X : \|x\| = \|y\| = 1, \|x - y\| > \epsilon \Rightarrow \frac{1}{2}\|x + y\| < 1 - \delta.$$ 

The goal of this project is to prove that such spaces satisfy a minimal norm property. Do the following.

(a) (Warm-up!) Show that any inner product space is uniformly convex.

(b) Show that if $X$ is uniformly convex and $(x_n) \subset X$ such that $\lim_{n \to \infty} \|x_n\| = 1$ and $\lim_{n, m \to \infty} \|x_n + x_m\| = 2$, then $(x_n)$ is convergent.

Proceed as follows. Use the definition to prove first that if all terms in the sequence have unit length, then the sequence is Cauchy. In the general case, let $y_n = x_n/\|x_n\|$. Prove that $\lim_{n, m \to \infty} \|y_n + y_m\| = 2$. Then, combine this with what you have shown already to prove that $(x_n)$ must be Cauchy.

(c) Prove that if $X$ is uniformly convex and $S \subset X$ is closed and convex, then $S$ contains a unique element of minimal norm, that is, a unique $s_0 \in S$ such that $\inf_{s \in S} \|s\| = \|s_0\|$.

Proceed as follows. Let $d = \inf_{s \in S} \|s\|$. Now, use part (b) to construct a sequence that converges to $d$. 

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