Project II: Reflexive spaces

**Definition 1.** Let $X$ be a normed space and $X^*$ its dual, endowed with the norm

$$
\|F\| = \sup_{\|x\| \leq 1} |F(x)|.
$$

We can define the embedding in the bidual $J_X : X \to (X^*)^*$ as

$$
X \ni x \mapsto J_X(x) \in X^{**},
$$

$$
J_X(x)(f) = f(x), \quad \forall f \in X^*.
$$

1. Prove that $J_X$ is a linear and injective transformation. In fact, $J_X$ is also an isometry. This fact can be proved using the so-called Hahn-Banach principle, fundamental to functional analysis (go to Greg Kubitz’s colloquium to learn more about it!). In particular, this principle implies that if $x \neq 0$ then there exists some $f_x \in X^*$ such that $f_x(x) = \|x\|$. You can use this fact to prove 1. Now, by identifying $X \equiv J_X(X)$, we can write $X \subset X^{**}$, effectively embedding the normed space $X$ in its bidual.

**Definition 2.** We say that the normed space $X$ is reflexive if the embedding $J_X : X \to X^{**}$ is a surjection. Note that since the dual of any normed space is Banach, any reflexive space must be a Banach space.

2. Show that if $X$ is reflexive, then $X^*$ is also reflexive.

**Hint:** We know that $J_X(X) = X^{**}$, and $J_{X^*}(X^*) \subset (X^{**})^*$. We need to show that the last inclusion is, in fact, an equality. Start with $\phi \in (X^{**})^*$. Construct $f \in X^*$, $f(x) = \phi(J_X(x))$. Are $\phi$ and $f$ related through $J_{X^*}$?

3. Any Hilbert space $H$ is reflexive.

**Hint:** Recall that $H^*$ is itself a Hilbert space with the inner-product

$$
[f, g] = (\psi_H(f), \psi_H(g)),
$$

where, for $f \in H^*$, $\psi_H(f) \in H$ is given by the Riesz representation theorem through $f(x) = (x, \psi_H(f))$ for all $x \in H$. We want to show that $J_H$ is a surjection. Pick a $\phi \in H^{**}$. Let $x = (\psi_H \circ \psi_{H^*})(\phi)$. Where does $x$ live? What’s the connection you can make?