instead of the plane wave. This is a horizontal two-dimensional plane wave whose amplitude exponentially decays in the \(-z\)-direction. It is called an evanescent wave (exponentially decaying in the \(-z\)-direction).

The continuity of \(u\) and \(\frac{\partial u}{\partial z}\) at \(z = 0\) is satisfied if

\[
1 + R = T \quad (4.6.19)
\]

\[
\frac{ik\lambda}{1-R} = Tk_1 \sqrt{\sin^2 \theta_1 - \frac{c^2}{c^2}}. \quad (4.6.20)
\]

These equations can be simplified and solved for the reflection coefficient and \(T\) (the amplitude of the evanescent wave at \(z = 0\)). \(R\) (and \(T\)) will be complex, corresponding to phase shifts of the reflected (and evanescent) wave.

**EXERCISES 4.6**

4.6.1. Show that for a plane wave given by (4.6.2), the number of waves in \(2\pi\) distance in the direction of the wave (the \(k\)-direction) is \(k \equiv |k|\).

4.6.2. Show that the phase of a plane wave stays the same moving in the direction of the wave if the velocity is \(v\).

4.6.3. In optics, the index of refraction is defined as \(n = \frac{c_{\text{light}}}{c}\). Express Snell's law using the indices of refraction.

4.6.4. Find \(R\) and \(T\) for the evanescent wave by solving the simultaneous equations (4.6.19) and (4.6.20).

4.6.5. Find \(R\) and \(T\) by assuming that \(k_3 = \pm i\beta\), where \(\beta\) is defined by (4.6.16). Which sign do we use to obtain exponential decay as \(z \to -\infty\)?

### Chapter 5

**Sturm-Liouville Eigenvalue Problems**

#### 5.1 Introduction

We have found the method of separation of variables to be quite successful in solving some homogeneous partial differential equations with homogeneous boundary conditions. In all examples we have analyzed so far the boundary value problem that determines the needed eigenvalues (separation constants) has involved the simple ordinary differential equation

\[
\frac{d^2 \phi}{dx^2} + \lambda \phi = 0. \tag{5.1.1}
\]

Explicit solutions of this equation determined the eigenvalues \(\lambda\) from the homogeneous boundary conditions. The principle of superposition resulted in our needing to analyze infinite series. We pursued three different cases (depending on the boundary conditions): Fourier sine series, Fourier cosine series, and Fourier series (both sines and cosines). Fortunately, we verified by explicit integration that the eigenfunctions were orthogonal. This enabled us to determine the coefficients of the infinite series from the remaining nonhomogeneous condition.

In this section we further explain and generalize these results. We show that the orthogonality of the eigenfunctions can be derived even if we cannot solve the defining differential equation in terms of elementary functions [as in (5.1.1)]. Instead, orthogonality is a direct result of the differential equation. We investigate other boundary value problems resulting from separation of variables that yield other families of orthogonal functions. These generalizations of Fourier series will not always involve sines and cosines since (5.1.1) is not necessarily appropriate in every situation.
5.2 Examples

5.2.1 Heat Flow in a Nonuniform Rod

In Sec. 1.2 we showed that the temperature $u$ in a nonuniform rod solves the following partial differential equation:

$$
\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( K(x) \frac{\partial u}{\partial x} \right) = Q(x,t),
$$

(5.2.1)

where $Q$ represents any possible sources of heat energy. Here, in order to consider the case of a nonuniform rod, we allow the thermal coefficients $c, \rho, K(x)$ to depend on $x$. The method of separation of variables can be applied only if (5.2.1) is linear and homogeneous. Usually, to make (5.2.1) homogeneous, we consider only situations without sources. $Q = 0$. However, we will be slightly more general. We will allow the heat source $Q$ to be proportional to the temperature $u$,

$$
Q = au,
$$

(5.2.2)

in which case

$$
\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) + auu = 0.
$$

(5.2.3)

We also allow $a$ to depend on $x$ (but not on $t$), as though the specific types of sources depend on the material. Although $Q \neq 0$, (5.2.3) is still a linear and homogeneous partial differential equation. To understand the effect of this source $Q$, we present a plausible physical situation in which terms such as $a(x)u$ might arise. Suppose that a chemical reaction generates heat (called an endothermic reaction) corresponding to $Q > 0$. Conceivably, this reaction could be more intense at higher temperatures. In this way the heat energy generated might be proportional to the temperature and thus $a > 0$ (assuming that $u > 0$). Other types of chemical reactions (known as exothermic) would remove heat energy from the rod and also could be proportional to the temperature. For positive temperatures ($u > 0$), this corresponds to $a < 0$. In our problem $a = a(x)$, and hence it is possible that $a > 0$ in some parts of the rod and $a < 0$ in other parts. We summarize these results by noting that if $a(x) < 0$ for all $x$, then heat energy is being taken out of the rod, and vice versa. Later in our mathematical analysis, we will correspondingly discuss the special case $a(x) < 0$.

Equation (5.2.3) is suited for the method of separation of variables if, in addition, we assume that there is one homogeneous boundary condition (as yet unspecified) at each end, $x = 0$ and $x = L$. We have already analyzed cases in which $a = 0$ and $c, \rho, K_0$ are constant. In separating variables, we substitute the product form

$$
u(x,t) = \phi(x)h(t),
$$

(5.2.4)

into (5.2.3), which yields

$$
cp\phi(x) \frac{dh}{dt} = h(t) \left( K_0 \frac{d^2 \phi}{dx^2} \right) + \alpha \phi(x)h(t).
$$

Dividing by $\phi(x)h(t)$ does not necessarily separate variables since $cp$ may depend on $x$. However, dividing by $cp\phi(x)h(t)$ is always successful:

$$
\frac{1}{h} \frac{dh}{dt} = \frac{1}{cp \phi} \frac{d}{dx} \left( K_0 \frac{d \phi}{dx} \right) + \frac{\alpha}{cp} = -\lambda.
$$

(5.2.5)

The separation constant $-\lambda$ has been introduced with a minus sign because in this form the time-dependent equation (following from (5.2.5)),

$$
\frac{dh}{dt} = -\lambda h,
$$

(5.2.6)

has exponentially decaying solutions if $\lambda > 0$. Solutions to (5.2.6) exponentially grow if $\lambda < 0$ (and $\lambda = 0$). Solutions exponentially growing in time are not usually encountered in physical problems. However, for problems in which $\lambda > 0$ for at least part of the rod, thermal energy is being put into the rod by the exothermic reaction, and hence it is possible for there to be some negative eigenvalues ($\lambda < 0$).

The spatial differential equation implied by separation of variables is

$$
\frac{d}{dx} \left( K_0 \frac{d \phi}{dx} \right) + \alpha \phi + \lambda cp \phi = 0,
$$

(5.2.7)

which forms a boundary value problem when complemented by two homogeneous boundary conditions. This differential equation is not $d^2 \phi/dx^2 + \lambda \phi = 0$. Neither does (5.2.7) have constant coefficients, because the thermal coefficients $K_0, c, \rho, \alpha$ are not constant. In general, one way in which nonconstant-coefficient differential equations occur is in situations where physical properties are nonuniform.

Note that we cannot decide on the appropriate convenient sign for the separation constant by quickly analyzing the spatial ordinary differential equation (5.2.7) with its homogeneous boundary conditions. Usually we will be unable to solve (5.2.7) in the variable coefficient case, other than by a numerical approximate solution on the computer. Consequently, we will describe in Sec. 5.3 certain important qualitative properties of the solution of (5.2.7). Later, with a greater understanding of (5.2.7), we will return to reinvestigate heat flow in a nonuniform rod. For now, let us describe another example that yields a boundary value problem with nonconstant coefficients.

5.2.2 Circularly Symmetric Heat Flow

Nonconstant-coefficient differential equations can also arise if the physical parameters are constant. In Sec. 1.5 we showed that if the temperature $u$ is some plane
two-dimensional region is circularly symmetric (so that \( u \) only depends on time \( t \) and on the radial distance \( r \) from the origin), then \( u \) solves the linear and homogeneous partial differential equation

\[
\frac{\partial u}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right),
\]

under the assumption that all the thermal coefficients are constant.

We apply the method of separation of variables by seeking solutions in the form of a product:

\[ u(r, t) = \phi(r)h(t). \]

Equation (5.2.8) then implies that

\[ \frac{d}{dt} \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) \right) = -\lambda. \]

The two ordinary differential equations implied by (5.2.9) are

\[ \frac{d\phi}{dr} + \lambda r \phi = 0. \]

The separation constant is denoted \(-\lambda\) since we expect solutions to exponentially decay in time, as is implied by (5.2.10) if \( \lambda > 0 \). The nonconstant coefficients in (5.2.11) are due to geometric factors introduced by the use of polar coordinates. Later in this text (Sec. 7.7) we will show that (5.2.11) can be solved using Bessel functions. However, the general discussions in the remainder of this chapter will be quite valuable in our understanding of this problem.

5.3 Sturm-Liouville Eigenvalue Problems

5.3.1 General Classification

Differential equation. A boundary value problem consists of a linear homogeneous differential equation and corresponding linear homogeneous boundary conditions. All of the differential equations for boundary value problems that have been formulated in this text can be put in the following form:

\[
\frac{d}{dx} \left( p \frac{d\phi}{dx} \right) + q \phi + \lambda \phi = 0,
\]

where \( \lambda \) is the eigenvalue. Here the variable \( x \) is defined on a finite interval \( a < x < b \). Four examples are as follows:

1. Simplest case: \( \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \); in which case, \( p = 1, q = 0, \sigma = 1 \).

2. Heat flow in a nonuniform rod: \( \frac{d}{dx} \left( K \frac{d\phi}{dx} \right) + a \phi + \lambda \sigma \phi = 0 \); in which case, \( p = K, q = a, \sigma = \epsilon \).

Both of these boundary conditions are exactly of the type we have already studied. However, for the circle, the same second-order differential equation (5.2.11) has only one boundary condition, \( u(b, t) = 0 \). Since the physical variable \( r \) ranges from \( r = 0 \) to \( r = b \), we need a homogeneous boundary condition at \( r = 0 \) for mathematical reasons. (This is the same problem that occurred in studying Laplace’s equation inside a cylinder. However, in that situation a nonhomogeneous condition was given at \( r = b \).) On the basis of physical reasoning, we expect that the condition at \( r = 0 \) is that the temperature is bounded there, \( |u(0, t)| < \infty \). This is an example of a singularity condition. It is homogeneous; it is the boundary condition that we apply at \( r = 0 \). Thus, we have homogeneous conditions at both \( r = 0 \) and \( r = b \) for the circle.
3. Vibrations of a nonuniform string. $T_0 \frac{d^2 \phi}{dt^2} + \alpha \phi + \lambda_0 \phi = 0$; in which case, $p = T_0$ (constant), $q = \alpha$, $\sigma = \rho_0$ (see Exercise 5.3.1).

4. Circularly symmetric heat flow: $\frac{d}{dr} \left( r \frac{d \phi}{dr} \right) + \lambda r \phi = 0$, here the independent variable $z = r$ and $p(x) = x$, $q(x) = 0$, $\sigma(x) = x$.

Many interesting results are known concerning any equation in the form (5.3.1). Equation (5.3.1) is known as a Sturm-Liouville differential equation, named after two famous mathematicians active in the mid-1800s who studied it.

**Boundary conditions.** The linear homogeneous boundary conditions that we have studied are of the form to follow. We also introduce some mathematical terminology:

<table>
<thead>
<tr>
<th>Heat flow</th>
<th>Vibrating string</th>
<th>Mathematical terminology</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi = 0$</td>
<td>Fixed (zero) temperature</td>
<td>First kind or Dirichlet condition</td>
</tr>
<tr>
<td>$\frac{d\phi}{dx} = 0$</td>
<td>Insulated</td>
<td>Second kind or Neumann condition</td>
</tr>
<tr>
<td>$\frac{d\phi}{dx}$</td>
<td>$\pm h \phi$</td>
<td>Third kind or Robin condition</td>
</tr>
</tbody>
</table>

- (Homogeneous) 
- Newton's law of cooling $0^\circ$ outside temperature, $h = k/T_0$, $h > 0$, $h > 0$ (physical)

- $\phi(-L) = \phi(L)$ Perfect thermal contact
- $\phi(0) < \infty$ Bounded temperature

| $\phi(0)$ | $\infty$ |
| $\phi(L)$ | $\infty$ |

5.3.2 Regular Sturm-Liouville Eigenvalue Problem

A regular Sturm-Liouville eigenvalue problem consists of the Sturm-Liouville differential equation,

$$\frac{d}{dx} \left( p(x) \frac{d \phi}{dx} \right) + q(x) \phi + \lambda \sigma(x) \phi = 0 \quad a < x < b,$$

subject to the boundary conditions that we have discussed (excluding periodic and singular cases):

\[
\begin{align*}
\beta_1 \phi(a) + \beta_2 \frac{d \phi}{da}(a) &= 0 \\
\beta_3 \phi(b) + \beta_4 \frac{d \phi}{db}(b) &= 0.
\end{align*}
\]

where $\beta_i$ are real. In addition, to be called regular, the coefficients $p, q,$ and $\sigma$ must be real and continuous everywhere (including the end points) and $p > 0$ and $\sigma > 0$ everywhere (also including the endpoints). For the regular Sturm-Liouville eigenvalue problem, many important general theorems exist. In Sec. 5.5 we will prove these results, and in Secs. 5.7 and 5.8 we will develop some more interesting examples that illustrate the significance of the general theorems.

**Statement of theorems.** At first let us just state (in one place) all the theorems we will discuss more fully later (and in some cases prove). For any regular Sturm-Liouville problem, all of the following theorems are valid:

1. All the eigenvalues $\lambda$ are real.
2. There exist an infinite number of eigenvalues:
   \[ \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots \]
   a. There is a smallest eigenvalue usually denoted $\lambda_1$.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue $\lambda_n$, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
   \[ f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x). \]
   Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients $a_n$ are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
   \[ \int_a^b \phi_n(x) \phi_m(x) \sigma(x) \, dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m. \]
6. Any eigenvalue can be related to its eigenfunction by the Rayleigh quotient:
   \[ \lambda = -p(x) \frac{d \phi}{dx} \sigma(x) \phi \quad \text{subject to} \quad \int_a^b \phi \phi \sigma \, dx, \]
   where the boundary conditions may somewhat simplify this expression.
5.3.3 Example and Illustration of Theorems

We will individually illustrate the meaning of these theorems (before proving many of them in Sec. 5.5) by referring to the simplest example of a regular Sturm-Liouville problem:

\[ \begin{align*}
\frac{d^2 \phi}{dx^2} + \lambda \phi &= 0, \\
\phi(0) &= 0, \\
\phi(L) &= 0.
\end{align*} \tag{5.3.4} \]

The constant-coefficient differential equation has zero boundary conditions at both ends. As we already know, the eigenvalues and corresponding eigenfunctions are

\[ \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{with} \quad \phi_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots, \]

giving rise to a Fourier sine series.

1. Real eigenvalues. Our theorem claims that all eigenvalues \( \lambda \) of a regular Sturm-Liouville problem are real. Thus, the eigenvalues of (5.3.4) should all be real. We know that the eigenvalues are \((n\pi/L)^2, n = 1, 2, \ldots\). However, in determining this result (see Sec. 2.3.4) we analyzed three cases: \( \lambda > 0, \lambda = 0, \) and \( \lambda < 0 \). We did not bother to look for complex eigenvalues because it is a relatively difficult task and we would have obtained no additional eigenvalues other than \((n\pi/L)^2\). This theorem (see Sec. 5.5 for its proof) is thus very useful. It guarantees that we do not even have to consider \( \lambda \) being complex.

2. Ordering of eigenvalues. There is an infinite number of eigenvalues for (5.3.4), namely \( \lambda = (n\pi/L)^2 \) for \( n = 1, 2, 3, \ldots \). Sometimes we use the notation \( \lambda_n = (n\pi/L)^2 \). Note that there is a smallest eigenvalue, \( \lambda_1 = (\pi/L)^2 \), but no largest eigenvalue since \( \lambda_n \to \infty \) as \( n \to \infty \). Our theorem claims that this idea is valid for any regular Sturm-Liouville problem.

3. Zeros of eigenfunctions. For the eigenvalues of (5.3.4), \( \lambda_n = (n\pi/L)^2 \), the eigenfunctions are known to be \( \sin n\pi x/L \). We use the notation \( \phi_n(x) = \sin n\pi x/L \).

An important and interesting aspect of this theorem is that we claim that for all regular Sturm-Liouville problems, the \( n \)th eigenfunction has exactly \((n - 1)\) zeros, not counting the endpoints. The eigenfunction \( \phi_1 \) corresponding to the smallest eigenvalue \( \lambda_1, n = 1 \) should have no zeros in the interior. The eigenfunction \( \phi_2 \) corresponding to the next smallest eigenvalue \( \lambda_2, n = 2 \) should have exactly one zero in the interior; and so on. We use our eigenvalue problem (5.3.4) to illustrate these properties. The eigenfunctions \( \phi_n(x) = \sin n\pi x/L \) are sketched in Fig. 5.3.1 for \( n = 1, 2, 3 \). Note that the theorem is verified (since we only count zeros at interior points); \( \sin \pi x/L \) has no zeros between \( x = 0 \) and \( x = L \), \( \sin 2\pi x/L \) has one zero between \( x = 0 \) and \( x = L \), and \( \sin 3\pi x/L \) has two zeros between \( x = 0 \) and \( x = L \).

4. Series of eigenfunctions. According to this theorem, the eigenfunctions can always be used to represent any piecewise smooth function \( f(x) \).

\[ f(x) \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}. \tag{5.3.5} \]

Thus, for our example (5.3.4),

\[ f(x) \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}. \]

We recognize this as a Fourier sine series. We know that any piecewise smooth function can be represented by a Fourier sine series and the infinite series converges to \( f(x) \) for \( 0 < x < L \). It converges to \( f(x) \) for \( x = 0 \) if \( f(x) \) is continuous there. This theorem thus claims that the convergence properties of Fourier sine series are valid for all series of eigenfunctions of any regular Sturm-Liouville eigenvalue problem. Equation (5.3.5) is referred to as an expansion of \( f(x) \) in terms of the eigenfunctions \( \phi_n(x) \) or, more simply, as an eigenfunction expansion. It is also called a generalized Fourier series of \( f(x) \). The coefficients \( a_n \) are called the coefficients of the eigenfunction expansion or the generalized Fourier coefficients. The fact that rather arbitrary functions may be represented in terms of an infinite series of eigenfunctions will enable us to solve partial differential equations by the method of separation of variables.
5. Orthogonality of eigenfunctions. The preceding theorem enables a function to be represented by a series of eigenfunctions, (5.3.5). Here we will show how to determine the generalized Fourier coefficients, $a_n$. According to the important theorem we are now describing, the eigenfunctions of any regular Sturm-Liouville eigenvalue problem will always be orthogonal. The theorem states that a weight $O''(x)$ equal to the coefficient of the right-hand side vanishes except when $n = m$, as we already know.

$$
\int_a^b \phi_n(x)\phi_m(x)\sigma(x) \, dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m. \tag{5.3.6}
$$

Here $\sigma(x)$ is the possibly variable coefficient that multiplies the eigenvalue $\lambda$ in the differential equation defining the eigenvalue problem. Since corresponding to each eigenvalue there is only one eigenfunction, the statement "if $\lambda_n \neq \lambda_m$" in (5.3.6) may be replaced by "if $n \neq m$." For the Fourier sine series example, the defining differential equation is $d^2q/dx^2 + \lambda q = 0$, and hence a comparison with the form of the general Sturm-Liouville problem shows that $\sigma(x) = 1$. Thus, in this case the weight is 1, and the orthogonality condition, $\int_a^b \sin nx/L \sin mx/L \, dx = 0$, follows if $n \neq m$, as we already know.

As with Fourier sine series, we use the orthogonality condition to determine the generalized Fourier coefficients. In order to utilize the orthogonality condition (5.3.6), we must multiply (5.3.5) by $\phi_m(x)$ and $\sigma(x)$. Thus,

$$
f(x) \phi_m(x) \sigma(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \phi_m(x) \sigma(x),
$$

where we assume these operations on infinite series are valid, and hence introduce equal signs. Integrating from $x = a$ to $x = b$ yields

$$
\int_a^b f(x) \phi_m(x) \sigma(x) \, dx = \sum_{n=1}^{\infty} a_n \int_a^b \phi_n(x) \phi_m(x) \sigma(x) \, dx.
$$

Since the eigenfunctions are orthogonal [with weight $\sigma(x)$], all the integrals on the right-hand side vanish except when $n$ reaches $m$:

$$
\int_a^b f(x) \phi_m(x) \sigma(x) \, dx = a_m \int_a^b \phi_m^2(x) \sigma(x) \, dx.
$$

The integral on the right is nonzero since the weight $\sigma(x)$ must be positive (from the definition of a regular Sturm-Liouville problem), and hence we may divide by it to determine the generalized Fourier coefficient $a_m$:

$$
a_m = \frac{\int_a^b f(x) \phi_m(x) \sigma(x) \, dx}{\int_a^b \phi_m^2(x) \sigma(x) \, dx}. \tag{5.3.7}
$$

6. Rayleigh quotient. In Sec. 5.6 we will prove that the eigenvalue may be related to its eigenfunction in the following way:

$$
\lambda = \frac{\int_a^b f(x) \phi(x) \sigma(x) \, dx}{\int_a^b \phi^2 \sigma \, dx}, \tag{5.3.8}
$$

known as the Rayleigh quotient. The numerator contains integrated terms and terms evaluated at the boundaries. Since the eigenfunctions cannot be determined without knowing the eigenvalues, this expression is never used directly to determine the eigenvalues. However, interesting and significant results can be obtained from the Rayleigh quotient without solving the differential equation. Consider the Fourier sine series example (5.3.4) that we have been analyzing: $a = 0, b = L, \phi(x) = 1, \sigma(x) = 0$, and $\sigma(x) = 1$. Since $\phi(0) = 0$ and $\phi(L) = 0$, the Rayleigh quotient implies that

$$
\lambda = \frac{\int_0^L (-\phi'(x)^2 + 1) \, dx}{\int_0^L \phi^2 \, dx}. \tag{5.3.9}
$$

Although this does not determine $\lambda$ since $\phi$ is unknown, it gives useful information. Both the numerator and the denominator are $\geq 0$. Since $\phi$ cannot be identically zero and be called an eigenfunction, the denominator cannot be zero. Thus, $\lambda \geq 0$ follows from (5.3.9).

Without solving the differential equation, we immediately conclude that there cannot be any negative eigenvalues. When we first determined eigenvalues for this problem, we worked rather hard to show that there were no negative eigenvalues (see Sec. 2.3). Now we can simply apply the Rayleigh quotient to eliminate the possibility of negative eigenvalues for this example. Sometimes, as we shall see later, we can also show that $\lambda > 0$ in harder problems.

Furthermore, even the possibility of $\lambda = 0$ can sometimes be analyzed using the Rayleigh quotient. For the simple problem (5.3.4) with zero boundary conditions, $\phi(0) = 0$ and $\phi(L) = 0$, let us see if it is possible for $\lambda = 0$ directly from (5.3.9). $\lambda = 0$ only if $\phi'(x)^2 = 0$ for all $x$. Then, by integration, $\phi$ must be a constant for all $x$. However, from the boundary conditions (either $\phi(0) = 0$ or $\phi(L) = 0$), that constant must be zero. Thus, $\lambda = 0$ only if $\phi = 0$ everywhere. But if $\phi = 0$ everywhere, we do not call $\phi$ an eigenfunction. Thus, $\lambda = 0$ is not an eigenvalue in this case, and we have further concluded that $\lambda > 0$: all the eigenvalues must be positive. This is concluded without using solutions of the differential equation. The known eigenvalues in this example, $\lambda_n = (n\pi/L)^2, n = 1, 2, \ldots$, are clearly consistent with the conclusions from the Rayleigh quotient. Other applications of the Rayleigh quotient will appear in later sections.
EXERCISES 5.3

*5.3.1. Do Exercise 4.4.2(b). Show that the partial differential equation may be put into Sturm-Liouville form.

5.3.2. Consider

\[ \rho \frac{\partial^2 u}{\partial t^2} + \alpha u + \beta \frac{\partial u}{\partial t} = T_0 \frac{\partial^2 u}{\partial x^2} \]

(a) Give a brief physical interpretation. What signs must \( \alpha \) and \( \beta \) have to be physical?

(b) Allow \( \rho, \alpha, \beta \) to be functions of \( x \). Show that separation of variables works only if \( \beta = c \alpha \), where \( c \) is a constant.

(c) If \( \beta = c \alpha \), show that the spatial equation is a Sturm-Liouville differential equation. Solve the time equation.

*5.3.3. Consider the non-Sturm-Liouville differential equation

\[ \frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + \frac{\alpha(x)}{x} \frac{d\phi}{dx} + [\lambda \beta(x) + \gamma(x)]\phi = 0. \]

Multiply this equation by \( H(x) \). Determine \( H(x) \) such that the equation may be reduced to the standard Sturm-Liouville form:

\[ \frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + [\lambda \beta(x) + \gamma(x)]\phi = 0. \]

Given \( \alpha(x), \beta(x), \) and \( \gamma(x) \), what are \( p(x), \sigma(x), \) and \( q(x) \)?

5.3.4. Consider heat flow with convection (see Exercise 1.5.2):

\[ \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + V \frac{\partial u}{\partial x}. \]

(a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm-Liouville form.

(b) Show that multiplying by \( 1/x \) puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

*(c) Solve the initial boundary value problem

\[ u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x). \]

(c) Solve the initial boundary value problem

\[ \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad u(x,0) = f(x). \]

5.3.5. For the Sturm-Liouville eigenvalue problem,

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0, \]

verify the following general properties:

(a) There is an infinite number of eigenvalues with a smallest but no largest.

(b) The \( n \)th eigenfunction has \( n - 1 \) zeros.

(c) The eigenfunctions are complete and orthogonal.

(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

5.3.6. Redo Exercise 5.3.5 for the Sturm-Liouville eigenvalue problem

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0. \]

5.3.7. Which of statements 1-5 of the theorems of this section are valid for the following eigenvalue problem?

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0. \]

5.3.8. Show that \( \frac{d\phi}{dx}(0) = 0 \) for the eigenvalue problem

\[ \frac{d^2 \phi}{dx^2} + \alpha(x) \frac{d\phi}{dx} + \beta(x) \phi = 0. \]

5.3.9. Consider the eigenvalue problem

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0. \]

5.3.10. Redo Exercise 5.3.9 with the boundary conditions

\[ \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(b) = 0. \]
5.4 Worked Example: Heat Flow in a Nonuniform Rod without Sources

In this section we illustrate the application to partial differential equations of some of the general theorems on regular Sturm-Liouville eigenvalue problems. Consider the heat flow in a nonuniform rod (with possibly nonconstant thermal properties \( c, \rho, K_0 \)) without sources; see Sec. 1.2 or 5.2.1. At the left end \( x = 0 \) the temperature is prescribed to be 0° and the right end is insulated. The initial temperature distribution is given. The mathematical formulation of this problem is

\[
PDE: \quad c u_t = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) \tag{5.4.1}
\]

BC: \[ u(0, t) = 0 \quad \frac{\partial u}{\partial x}(L, t) = 0 \tag{5.4.2} \]

IC: \[ u(x, 0) = f(x). \tag{5.4.3} \]

Since the partial differential equation and the boundary conditions are linear and homogeneous, we seek special solutions (ignoring the initial condition) in the product form:

\[ u(x, t) = \phi(x)h(t). \tag{5.4.4} \]

After separation of variables (for details see Sec. 5.2.1), we find that the time part satisfies the ordinary differential equation

\[ \frac{dh}{dt} = -\lambda h, \tag{5.4.5} \]

while the spatial part solves the following regular Sturm-Liouville eigenvalue problem:

\[ \frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) + \lambda c \rho \phi = 0 \tag{5.4.6} \]

\[ \phi(0) = 0 \tag{5.4.7} \]

\[ \frac{d\phi}{dx}(L) = 0. \tag{5.4.8} \]

According to our theorems concerning Sturm-Liouville eigenvalue problems, there is an infinite sequence of eigenvalues \( \lambda_n \) and corresponding eigenfunctions \( \phi_n(x) \). We assume that \( \phi_n(x) \) are known (it might be a difficult problem to determine approximately the first few using numerical methods, but nevertheless it can be done). The time-dependent part of the differential equation is easily solved,

\[ h(t) = c e^{-\lambda t}. \tag{5.4.9} \]

In this way we obtain an infinite sequence of product solutions of the partial differential equation

\[ u(x, t) = \phi_n(x) e^{-\lambda_n t}. \tag{5.4.10} \]

According to the principle of superposition, we attempt to satisfy the initial condition with an infinite linear combination of these product solutions:

\[ u(x, t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}. \tag{5.4.11} \]

This infinite series has the property that it solves the PDE and the homogeneous BCs. We will show that we can determine the as yet unknown constants \( a_n \) from the initial condition

\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{5.4.12} \]

Our theorems imply that any piecewise smooth \( f(x) \) can be represented by this type of series of eigenfunctions. The coefficients \( a_n \) are the generalized Fourier coefficients of the initial condition. Furthermore, the eigenfunctions are orthogonal with a weight \( \alpha(x) = c(x) \rho(x) \), determined from the physical properties of the rod:

\[ \int_0^L \phi_n(x) \phi_m(x) c(x) \rho(x) \, dx = 0 \quad \text{for} \quad n \neq m. \]

Using these orthogonality formulas, the generalized Fourier coefficients are

\[ a_n = \frac{\int_0^L f(x) \phi_n(x) c(x) \rho(x) \, dx}{\int_0^L \phi_n^2(x) c(x) \rho(x) \, dx}. \tag{5.4.13} \]

We claim that (5.4.11) is the desired solution, with coefficients given by (5.4.13).

In order to give a minimal interpretation of the solution, we should ask what happens for large \( t \). Since the eigenvalues form an increasing sequence, each succeeding term in (5.4.11) is exponentially smaller than the preceding term for large \( t \). Thus, for large time the solution may be accurately approximated by

\[ u(x, t) \approx a_1 \phi_1(x) e^{-\lambda_1 t}. \tag{5.4.14} \]
This approximation is not very good if \(a_1 = 0\), in which case (5.4.14) should begin with the first nonzero term. However, often the initial temperature \(f(x)\) is nonnegative (and not identically zero). In this case, we will show from (5.4.13) that \(a_1 \neq 0\):

\[
a_1 = \int_0^L f(x) \phi_1(x) c(x) \rho(x) \, dx - \int_0^L \phi_1'(x) c(x) \rho(x) \, dx.
\]  

(5.4.15)

It follows that \(a_1 \neq 0\), because \(\phi_1(x)\) is the eigenfunction corresponding to the lowest eigenvalue and has no zeros; \(\phi_1(x)\) is of one sign. Thus, if \(f(x) > 0\) it follows that \(a_1 \neq 0\), since \(c(x)\) and \(\rho(x)\) are positive physical functions. In order to sketch the solution for large fixed \(t\), (5.4.14) shows that all that is needed is the first eigenfunction. At the very least, a numerical calculation of the first eigenfunction is easier than the computation of the first hundred.

For large time, the “shape” of the temperature distribution in space stays approximately the same in time. Its amplitude grows or decays in time depending on whether \(a_1 > 0\), since the thermal coefficients are positive. Furthermore, \(a(x) = 0\), and \(\phi_1(x)\) is of one sign. Thus, if \(f(x) > 0\) it follows from the Rayleigh quotient that

\[
\int_0^L K_0(x)(\phi_1(x)c(x)\rho(x)) \, dx = a_1 \int_0^L K_0(x)(\phi_1(x)c(x)\rho(x)) \, dx.
\]

(5.4.16)

where the boundary contribution to (5.4.16) vanished due to the specific homogeneous boundary conditions, (5.4.7) and (5.4.8). It immediately follows from (5.4.16) that all \(\lambda > 0\), since the thermal coefficients are positive. Furthermore, \(\lambda > 0\), since \(\phi = \text{constant}\) is not an allowable eigenfunction because \(\phi(0) = 0\). Thus, we have shown that \(\lim_{t \to \infty} u(x,t) = 0\) for this example.

**EXERCISES 5.4**

5.4.1. Consider

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) + \alpha u,
\]

where \(c, \rho, K_0, \alpha\) are functions of \(x\), subject to

\[
\begin{align*}
u(0, t) &= 0 \\
u(L, t) &= 0 \\
u(x, 0) &= f(x).
\end{align*}
\]

Assume that the appropriate eigenfunctions are known.

(a) Show that the eigenvalues are positive if \(\alpha < 0\) (see Sec. 5.2.1).

(b) Solve the initial value problem.

(c) Briefly discuss \(\lim_{t \to \infty} u(x,t)\).

*5.4.2. Consider

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( c \frac{\partial u}{\partial x} \right) + \alpha u,
\]

where \(c, \rho, K_0\) are functions of \(x\), subject to

\[
\begin{align*}
u(0, t) &= 0 \\
u(L, t) &= 0 \\
u(x, 0) &= f(x).
\end{align*}
\]

Assume that the appropriate eigenfunctions are known. Solve the initial value problem. Briefly discussing \(\lim_{t \to \infty} u(x,t)\).

*5.4.3. Solve

\[
\frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right)
\]

with \(u(x, 0) = f(x), u(0, t) = 0\), and \(u(\alpha, t) = 0\). You may assume that the corresponding eigenfunctions, denoted \(\phi_n(r)\), are known and are complete. (Hint: See Sec. 5.2.2.)

*5.4.4. Consider the following boundary value problem:

\[
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0
\]

with \(u(x, 0) = \sin \pi x / L\) (initial condition). (Hint: If necessary, use a table of integrals.)

*5.4.5. Consider

\[
\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + au.
\]

where \(\rho(x) > 0, \alpha(x) < 0\), and \(T_0\) is constant, subject to

\[
\begin{align*}
u(0, t) &= 0 \\
u(L, t) &= 0 \\
u(x, 0) &= f(x).
\end{align*}
\]

Assume that the appropriate eigenfunctions are known. Solve the initial value problem.

*5.4.6. Consider the vibrations of a *nonuniform* string of mass density \(p_0(x)\). Suppose that the left end at \(x = 0\) is fixed and the right end obeys the elastic boundary condition: \(\partial u / \partial x = -(K_0 / \rho_0) u\) at \(x = L\). Suppose that the string is initially at rest with a known initial position \(f(x)\). Solve this initial value problem. (Hints: Assume that the appropriate eigenvalues and corresponding eigenfunctions are known. What differential equations with what boundary conditions do they satisfy? The eigenfunctions are orthogonal with what weighting function?)
5.5 Self-Adjoint Operators and Sturm-Liouville Eigenvalue Problems

Introduction. In this section we prove some of the properties of regular Sturm-Liouville eigenvalue problems:

\[
\frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + q(x)\phi + \lambda \sigma(x)\phi = 0 \tag{5.5.1}
\]

\[
\beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0 \tag{5.5.2}
\]

\[
\beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0, \tag{5.5.3}
\]

where \(\beta_i\) are real and where, on the finite interval \((a \leq x \leq b)\), \(p, q, \sigma\) are real continuous functions and \(p, q, \sigma\) are positive \([p(x) > 0\) and \(\sigma(x) > 0\)]. At times we will make some comments on the validity of our results if some of these restrictions are removed.

The proofs of three statements are somewhat difficult. We will not prove that there are an infinite number of eigenvalues. We will have to rely for understanding on the examples already presented and on some further examples developed in later sections. For Sturm-Liouville eigenvalue problems that are not regular, there may be no eigenvalues at all. However, in most cases of physical interest (on finite intervals) there will still be an infinite number of discrete eigenvalues. We also will not attempt to prove that any piecewise smooth function can be expanded in terms of the eigenfunctions of a regular Sturm-Liouville problem (known as the completeness property). We will not attempt to prove that each succeeding eigenfunction has one additional zero (oscillates one more time).

Linear operators. The proofs we will investigate are made easier to follow by the introduction of operator notation. Let \(L\) stand for the linear differential operator \(\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)\). An operator acts on a function and yields another function. The notation means that for this \(L\) acting on the function \(y(x)\),

\[
L(y) = \frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] + q(x)y. \tag{5.5.4}
\]

Thus, \(L(y)\) is just a shorthand notation. For example, if \(L = \frac{d^2}{dx^2} + 6\), then \(L(y) = \frac{d^2y}{dx^2} + 6y\) or \(L(e^{2x}) = 4e^{2x} + 6e^{2x} = 10e^{2x}\).

The Sturm-Liouville differential equation is rather cumbersome to write over and over again. The use of the linear operator notation is somewhat helpful. Using the operator notation

\[
L(\phi) + \lambda \sigma(x)\phi = 0, \tag{5.5.5}
\]

where \(\lambda\) is an eigenvalue and \(\phi\) the corresponding eigenfunction, \(L\) can operate on any function, not just an eigenfunction.

Lagrange’s identity. Most of the proofs we will present concerning Sturm-Liouville eigenvalue problems are immediate consequences of an interesting and fundamental formula known as Lagrange’s identity. For convenience, we will use the operator notation. We calculate \(uL(v) - vL(u)\), where \(u\) and \(v\) are any two functions (not necessarily eigenfunctions). Recall that

\[
L(u) = \frac{d}{dx} \left[ \frac{du}{dx} \right] + qu \quad \text{and} \quad L(v) = \frac{d}{dx} \left[ \frac{dv}{dx} \right] + qv,
\]

and hence

\[
uL(v) - vL(u) = \frac{d}{dx} \left[ \frac{du}{dx}v - v\frac{dv}{dx} \right] = \frac{dv}{dx}u - \frac{du}{dx}v.
\]

where a simple cancellation of \(uqv - vqu\) should be noted. The right-hand side of (5.5.6) is manipulated to an exact differential:

\[
\frac{dv}{dx}u - \frac{du}{dx}v = -\int_a^b \frac{d}{dx} \left[ \frac{du}{dx}v - v\frac{dv}{dx} \right] dx = -\int_a^b \frac{d}{dx} \left[ u\frac{dv}{dx} - v\frac{du}{dx} \right] dx
\]

known as the differential form of Lagrange’s identity. To derive (5.5.7), we note from the product rule that

\[
\frac{d}{dx} \left[ \frac{du}{dx}v - v\frac{dv}{dx} \right] = \frac{d}{dx} \left[ \frac{dv}{dx}u - \frac{du}{dx}v \right] - \frac{dv}{dx}u + \frac{du}{dx}v.
\]

and similarly

\[
\frac{d}{dx} \left[ u\frac{dv}{dx} - v\frac{du}{dx} \right] = \frac{d}{dx} \left[ \frac{dv}{dx}u - \frac{du}{dx}v \right] - \frac{dv}{dx}u + \frac{du}{dx}v.
\]

Equation (5.5.7) follows by subtracting these two. Later [see (5.5.21)] we will use the differential form, (5.5.7).

Green’s formula. The integral form of Lagrange’s identity is also known as Green’s formula. If follows by integrating (5.5.7):

\[
\int_a^b [uL(v) - vL(u)] dx = \int_a^b \left[ u\frac{dv}{dx} - v\frac{du}{dx} \right] dx
\]

for any functions, \(u\) and \(v\). This is a very useful formula.

\(1\)The integration requires \(du/dx\) and \(dv/dx\) to be continuous.
Example. If \( p = 1 \) and \( q = 0 \) (in which case \( L = d^2 / dx^2 \)), (5.5.7) simply states that
\[
\frac{d^2 u}{dx^2} - \frac{d^2 v}{dx^2} = \frac{d}{dx} \left( \frac{dv}{dx} - \frac{du}{dx} \right),
\]
which is easily independently checked. For this example, Green’s formula is
\[
\int_a^b \left( \frac{d^2 u}{dx^2} - \frac{d^2 v}{dx^2} \right) \, dx = \left( \frac{dv}{dx} - \frac{du}{dx} \right)_a^b.
\]

Self-adjointness. As an important case of Green’s formula, suppose that \( u \) and \( v \) are any two functions, but with the additional restriction that the boundary terms happen to vanish,
\[
p \left( \frac{dv}{dx} - \frac{du}{dx} \right)_a^b = 0.
\]
Then from (5.5.8), \( \int_a^b [uL(v) - vL(u)] \, dx = 0 \).

Let us show how it is possible for the boundary terms to vanish. Instead of being arbitrary functions, we restrict \( u \) and \( v \) to both satisfy the same set of homogeneous boundary conditions. For example, suppose that \( u \) and \( v \) are any two functions that satisfy the following set of boundary conditions:
\[
\begin{align*}
\phi(a) &= 0 \\
\frac{d \phi}{dx}(b) + h \phi(b) &= 0.
\end{align*}
\]
Since both \( u \) and \( v \) satisfy these conditions, it follows that
\[
\frac{d u}{dx}(b) + h u(b) = 0 \quad \text{and} \quad \frac{d v}{dx}(b) + h v(b) = 0;
\]
otherwise, \( u \) and \( v \) are arbitrary. In this case, the boundary terms for Green’s formula vanish:
\[
p \left( \frac{dv}{dx} - \frac{du}{dx} \right)_a^b = p(b) \left( v(b) \frac{du}{dx}(b) - u(b) \frac{dv}{dx}(b) \right)
= p(b) \left[ -u(b) h u(b) + v(b) h v(b) \right] = 0.
\]
Thus, for any functions \( u \) and \( v \) both satisfying these homogeneous boundary conditions, we know that
\[
\int_a^b [uL(v) - vL(u)] \, dx = 0.
\]
In fact, we claim (see Exercise 5.5.1) that the boundary terms also vanish for any two functions \( u \) and \( v \) that both satisfy the same set of boundary conditions of the type that occur in the regular Sturm-Liouville eigenvalue problems (5.5.2) and (5.5.3).

Orthogonal eigenfunctions. We now will show the usefulness of Green’s formula. We will begin by proving the important orthogonality relationship for Sturm-Liouville eigenvalue problems. For many types of boundary conditions, eigenfunctions corresponding to different eigenvalues are orthogonal with weight \( \sigma(x) \). To prove that statement, let \( \lambda_m \) and \( \lambda_n \) be eigenvalues with corresponding eigenfunctions \( \phi_m(x) \) and \( \phi_n(x) \). Using the operator notation, the differential equations satisfied by these eigenfunctions are
\[
L(\phi_m) + \lambda_m \sigma(x) \phi_m = 0 \\
L(\phi_n) + \lambda_n \sigma(x) \phi_n = 0.
\]
In addition, both \( \phi_m \) and \( \phi_n \) satisfy the same set of homogeneous boundary conditions. Since \( u \) and \( v \) are arbitrary functions, we may let \( u = \phi_m \) and \( v = \phi_n \) in Green’s formula:
\[
\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)] \, dx = p(x) \left( \phi_m \frac{d \phi_n}{dx} - \phi_n \frac{d \phi_m}{dx} \right)_a^b.
\]
Thus, when discussing any regular Sturm-Liouville eigenvalue problem, we have the following theorem:

If \( u \) and \( v \) are any two functions satisfying the same set of homogeneous boundary conditions (of the regular Sturm-Liouville type), then
\[
\int_a^b [uL(v) - vL(u)] \, dx = 0.
\]
When (5.5.9) is valid, we say that the operator \( L \) (with the corresponding boundary conditions) is self-adjoint.\(^3\)

The boundary terms also vanish in circumstances other than for boundary conditions of the regular Sturm-Liouville type. Two important further examples will be discussed briefly. The periodic boundary condition can be generalized (for nonconstant-coefficient operators) to
\[
\phi(a) = \phi(b) \quad \text{and} \quad p(a) \frac{d \phi}{dx}(a) = p(b) \frac{d \phi}{dx}(b).
\]
In this situation (5.5.9) also can be shown (see Exercise 5.5.1) to be valid. Another example in which the boundary terms in Green’s formula vanish is the “singular” case. The singular case occurs if the coefficient of the second derivative of the differential operator is zero at an endpoint; for example, if \( p(x) = 0 \) at \( x = 0 \) (i.e., \( p(0) = 0 \)). At a singular endpoint, a singularity condition is imposed. The usual singularity condition at \( x = 0 \) is \( \phi(0) \) bounded. It can also be shown that (5.5.9) is valid (see Exercise 5.5.1) if both \( u \) and \( v \) satisfy this singularity condition at \( x = 0 \) and any regular Sturm-Liouville type of boundary condition at \( x = b \).

\(^3\)We usually avoid in this text an explanation of an adjoint operator. Here \( L \) equals its adjoint and so is called self-adjoint.
$L(\phi_n)$ and $L(\phi_m)$ may be eliminated from (5.5.10) and (5.5.11). Thus,

$$\left(\lambda_m - \lambda_n\right) \int_a^b \phi_n \phi_m \sigma \, dx = p(x) \left( \phi_m \frac{d\phi_m}{dx} - \phi_n \frac{d\phi_n}{dx} \right),$$

(5.5.12)

corresponding to multiplying (5.5.10) by $\phi_m$, multiplying (5.5.11) by $\phi_n$, subtracting the two, and then integrating. We avoided these steps (especially the integration) by applying Green's formula. For many different kinds of boundary conditions (i.e., regular Sturm-Liouville types, periodic case, and the singular case), the boundary terms vanish if $u$ and $v$ both satisfy the same set of homogeneous boundary conditions. Since $u$ and $v$ are eigenfunctions, they satisfy this condition, and thus (5.5.12) implies that

$$\left(\lambda_m - \lambda_n\right) \int_a^b \phi_n \phi_m \sigma \, dx = 0.$$  

(5.5.13)

If $\lambda_m \neq \lambda_n$, then it immediately follows that

$$\int_a^b \phi_n \phi_m \sigma \, dx = 0.$$  

(5.5.14)

In other words, eigenfunctions ($\phi_n$ and $\phi_m$) corresponding to different eigenvalues ($\lambda_n \neq \lambda_m$) are orthogonal with weight $\sigma(x)$.

Real eigenvalues. We can use the orthogonality of eigenfunctions to prove that the eigenvalues are real. Suppose that $\lambda$ is a complex eigenvalue and $\phi(x)$ the corresponding eigenfunction (also allowed to be complex since the differential equation defining the eigenfunction would be complex):

$$L(\phi) + \lambda \sigma \phi = 0.$$  

(5.5.15)

We introduce the notation $\overline{\phi}$ for the complex conjugate (e.g., if $z = x + iy$, then $\overline{z} = x - iy$). Note that if $\phi = 0$, then $\overline{\phi} = 0$. Thus, the complex conjugate of (5.5.15) is also valid:

$$\overline{L(\phi)} + \overline{\lambda} \sigma \overline{\phi} = 0,$$  

(5.5.16)

assuming that the coefficient $\sigma$ is real and hence $\overline{\sigma} = \sigma$. The complex conjugate of $L(\phi)$ is exactly $L$ operating on the complex conjugate of $\phi$, $L(\overline{\phi}) = \overline{L(\phi)}$ since the coefficients of the linear differential operator are also real (see Exercise 5.5.7). Thus,

$$L(\overline{\phi}) + \overline{\lambda} \sigma \overline{\phi} = 0.$$  

(5.5.17)

If $\phi$ satisfies boundary conditions with real coefficients, then $\overline{\phi}$ satisfies the same boundary conditions. For example, if $d\phi/dx + h\phi = 0$ at $x = a$, then by taking complex conjugates, $d\overline{\phi}/dx + h\overline{\phi} = 0$ at $x = a$. Equation (5.5.17) and the boundary conditions show that $\overline{\phi}$ satisfies the Sturm-Liouville eigenvalue problem, but with the coefficient being $\overline{\lambda}$. We have thus proved the following theorem: If $\lambda$ is a complex eigenvalue with corresponding eigenfunction $\phi$, then $\overline{\lambda}$ is also an eigenvalue with corresponding eigenfunction $\overline{\phi}$.

However, we will show $\lambda$ cannot be complex. As we have shown, if $\lambda$ is an eigenvalue, then so too is $\overline{\lambda}$. According to our fundamental orthogonality theorem, the corresponding eigenfunctions ($\phi$ and $\overline{\phi}$) must be orthogonal (with weight $\sigma$). Thus, from (5.5.13),

$$\int_a^b \phi \overline{\phi} \sigma \, dx = 0.$$

(5.5.18)

Since $\overline{\phi} = |\phi|^2 \geq 0$ (and $\sigma > 0$), the integral in (5.5.18) is $\geq 0$. In fact, the integral can equal zero only if $\phi = 0$, which is prohibited since $\phi$ is an eigenfunction. Thus, (5.5.18) implies that $\lambda = \overline{\lambda}$, and hence $\lambda$ is real; all the eigenvalues are real. The eigenfunctions can always be chosen to be real.

Unique eigenfunctions (regular and singular cases). We next prove that there is only one eigenfunction corresponding to an eigenvalue (except for the case of periodic boundary conditions). Suppose that there are two different eigenfunctions $\phi_1$ and $\phi_2$ corresponding to the same eigenvalue $\lambda$. We say $\lambda$ is a "multiple" eigenvalue with multiplicity two. In this case, both

$$L(\phi_1) + \lambda \sigma \phi_1 = 0,$$

$$L(\phi_2) + \lambda \sigma \phi_2 = 0.$$  

(5.5.19)

Since $\lambda$ is the same in both expressions,

$$\phi_1 L(\phi_1) - \phi_1 L(\phi_2) = 0.$$  

(5.5.20)

This can be integrated by some simple manipulations. However, we avoid this algebra by simply quoting the differential form of Lagrange's identity:

$$\phi_1 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx} \left[ \phi_1 \frac{d\phi_1}{dx} - \phi_2 \frac{d\phi_2}{dx} \right].$$

(5.5.21)

From (5.5.20) it follows that

$$\frac{d}{dx} \left( \phi_1 \frac{d\phi_1}{dx} - \phi_2 \frac{d\phi_2}{dx} \right) = \text{constant}.$$  

(5.5.22)

Often we can evaluate the constant from one of the boundary conditions. For example, if $d\phi_1/dx + h\phi_1 = 0$ at $x = a$, a short calculation shows that the constant equals zero. In fact, we claim (Exercise 5.5.10) that the constant also equals zero if at least one of the boundary conditions is of the regular Sturm-Liouville type (or of the singular type). For any of these boundary conditions it follows that

$$\phi_1 \frac{d\phi_1}{dx} - \phi_2 \frac{d\phi_2}{dx} = 0.$$  

(5.5.23)

3A "similar" type of theorem follows from the quadratic formula: For a quadratic equation with real coefficients, if $\lambda$ is a complex root, then so is $\overline{\lambda}$. This also holds for any algebraic equation with real coefficients.

*Note:*
This is equivalent to \( \frac{d}{dx}(\phi_2/\phi_1) = 0 \), and hence for these boundary conditions

\[
\phi_2 = c\phi_1.
\]

(5.5.24)

This shows that any two eigenfunctions \( \phi_1 \) and \( \phi_2 \) corresponding to the same eigenvalue must be an integral multiple of each other for the preceding boundary conditions. The two eigenfunctions are dependent; there is only one linearly independent eigenfunction; the eigenfunction is unique.

**Nonunique eigenfunctions (periodic case).** For periodic boundary conditions, we cannot conclude that the constant in (5.5.22) must be zero. Thus, it is possible that \( \phi_2 \neq c\phi_1 \) and that there might be two different eigenfunctions corresponding to the same eigenvalue.

For example, consider the simple eigenvalue problem with periodic boundary conditions,

\[
\phi(-L) = \phi(L).
\]

(5.5.25)

We know that the eigenvalue 0 has any constant as the unique eigenfunction. The other eigenvalues, \( \lambda = n\pi/L \), \( n = 1, 2, \ldots \), each have two linearly independent eigenfunctions, \( \sin n\pi x/L \) and \( \cos n\pi x/L \). This, we know, gives rise to a Fourier series. However, (5.5.25) is not a regular Sturm-Liouville eigenvalue problem since the boundary conditions are not of the prescribed form. Our theorem about unique eigenfunctions does not apply; we may have two eigenfunctions corresponding to the same eigenvalue. Note that it is still possible to have only one eigenfunction, as occurs for \( \lambda = 0 \).

**Nonunique eigenfunctions (Gram-Schmidt orthogonalization).**

We can solve for generalized Fourier coefficients (and correspondingly we are able to solve some partial differential equations) because of the orthogonality of the eigenfunctions. However, our theorem states that eigenfunctions corresponding to different eigenvalues are automatically orthogonal [with weight \( \sigma(x) \)]. For the case of periodic (or mixed-type) boundary conditions, it is possible for there to be more than one independent eigenfunction corresponding to the same eigenvalue. For these multiple eigenvalues the eigenfunctions are not automatically orthogonal to each other. In the appendix to Sec. 7.5 we will show that we are always able to construct the eigenfunctions such that they are orthogonal by a process called Gram-Schmidt orthogonalization.

4No more than two independent eigenfunctions are possible, since the differential equation is of second order.

**EXERCISES 5.5**

5.5.1. A Sturm-Liouville eigenvalue problem is called self-adjoint if

\[
p \left( \frac{dv}{dx} - \frac{du}{dx} \right)_a^b = 0
\]

since then \( \int_a^b [uL(v) - vL(u)] \ dx = 0 \) for any two functions \( u \) and \( v \) satisfying the boundary conditions. Show that the following yield self-adjoint problems.

(a) \( \phi(0) = 0 \) and \( \phi(L) = 0 \)

(b) \( \phi'(0) = 0 \) and \( \phi(L) = 0 \)

(c) \( \phi''(0) - h\phi(0) = 0 \) and \( \phi''(L) = 0 \)

(d) \( \phi(a) = \phi(b) \) and \( p(a)\phi'(a) = p(b)\phi'(b) \)

(e) \( \phi(a) = \phi(b) \) and \( \phi''(a) = \phi''(b) \) [self-adjoint only if \( p(a) = p(b) \)]

(f) \( \phi(L) = 0 \) and \( \lim_{x \to a} p(x)\phi'(x) = 0 \)

(g) Under what conditions is the following self-adjoint (if \( p \) is constant)?

\[
\phi(L) + c\phi(0) + b\phi'(0) = 0
\]

(5.5.26)

5.5.2. Prove that the eigenfunctions corresponding to different eigenvalues (of the following eigenvalue problem) are orthogonal:

\[
\frac{d}{dx} \left( p(x)\phi' \right) + q(x)\phi + \lambda \sigma(x)\phi = 0
\]

with the boundary conditions

\[
\phi(1) = 0,
\]

\[
\phi(2) - 2\phi'(2) = 0.
\]

What is the weighting function?

5.5.3. Consider the eigenvalue problem \( L(\phi) = -\lambda \sigma(x)\phi \), subject to a given set of homogeneous boundary conditions. Suppose that

\[
\int_a^b [uL(v) - vL(u)] \ dx = 0
\]

for all functions \( u \) and \( v \) satisfying the same set of boundary conditions. Prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).
5.5.4. Give an example of an eigenvalue problem with more than one eigenfunction
    corresponding to an eigenvalue.

5.5.5. Consider
    \[ L = \frac{d^4}{dx^4} + 6 \frac{d}{dx} + 9. \]

(a) Show that \( L(e^{x^2}) = (r + 3)^2 e^{x^2}. \)

(b) Use part (a) to obtain solutions of \( L(y) = 0 \) (a second-order constant-coefficient differential equation).

(c) If \( z \) depends on \( x \) and a parameter \( r \), show that
    \[ \frac{\partial}{\partial r} L(z) = L \left( \frac{\partial z}{\partial r} \right). \]

(d) Using part (c), evaluate \( L(\partial z/\partial r) \) if \( z = e^{rx} \).

5.5.6. Prove that if \( x \) is a root of a sixth-order polynomial with real coefficients,
    then \( x \) is also a root.

5.5.7. For
    \[ L = \frac{d}{dx} \left( p \frac{d}{dx} \right) + q \]
    with \( p \) and \( q \) real, carefully show that
    \[ L(\varphi) = L(\tilde{\varphi}). \]

5.5.8. Consider a fourth-order linear differential operator,
    \[ L = \frac{d^4}{dx^4}. \]

(a) Show that \( uL(v) - vL(u) \) is an exact differential.

(b) Evaluate \( \int_a^b [uL(v) - vL(u)] \, dx \) in terms of the boundary data for any
    functions \( u \) and \( v \).

(c) Show that \( \int_a^b [uL(v) - vL(u)] \, dx = 0 \) if \( u \) and \( v \) are any two functions
    satisfying the boundary conditions
    \[ \phi(0) = 0, \quad \phi(1) = 0, \]
    \[ \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0. \]

(d) Give another example of boundary conditions such that
    \[ \int_a^b [uL(v) - vL(u)] \, dx = 0. \]

5.5.9. For the eigenvalue problem
    \[ \frac{d^4}{dx^4} \phi + \lambda e^x \phi = 0, \]
    show that the eigenfunctions corresponding to different eigenvalues are
    orthogonal. What is the weighting function?

5.5.10. (a) Show that \( (5.5.22) \) yields \( (5.5.23) \) if at least one of the boundary con-
    ditions is of the regular Sturm-Liouville type.

(b) Do part (a) if one boundary condition is of the singular type.

5.5.11. * Consider a fourth-order linear differential operator,
    \[ L = \frac{d^4}{dx^4} + \lambda e^x. \]

(a) Show that \( L = L^* \), the self-adjoint case? [Hint: Show that
    \[ L^* = \frac{d^4}{dx^4} + \left( \frac{d\phi}{dx} - r \right) \frac{d}{dx} + \left( \lambda e^x + \frac{dr}{dx} + q \right) \].

(b) If \( u(0) = 0 \) and \( \frac{du}{dx}(L) + u(L) = 0 \), what boundary conditions should \( u(x) \) satisfy for \( H(x) \) is called
    the adjoint boundary conditions?
5.5.12. Consider nonself-adjoint operators as in Exercise 5.5.11. The eigenvalues $\lambda$ may be complex as well as their corresponding eigenfunctions $\phi$.

(a) Show that if $\lambda$ is a complex eigenvalue with corresponding eigenfunction $\phi$, then the complex conjugate $\bar{\lambda}$ is also an eigenvalue with eigenfunction $\bar{\phi}$.

(b) The eigenvalues of the adjoint $L^*$ may be different from the eigenvalues of $L$. Using the result of Exercise 5.5.11, show that the eigenfunctions of $L(\phi) + \lambda\phi = 0$ are orthogonal with weight $\sigma$ (in a complex sense) to eigenfunctions of $L^*(\psi) + \bar{\lambda}\psi = 0$ if the eigenvalues are different. Assume that $\psi$ satisfies adjoint boundary conditions. You should also use part (a).

5.5.13. Using the result of Exercise 5.5.11, prove the following part of the Fredholm alternative (for operators that are not necessarily self-adjoint): A solution of $L(u) = f$ subject to homogeneous boundary conditions may exist only if $f(x)$ is orthogonal to all solutions of the homogeneous adjoint problem.

5.5.14. If $L$ is the following first-order linear differential operator

$$L = p(x) \frac{d}{dx},$$

then determine the adjoint operator $L^*$ such that

$$\int_a^b [uL^*(v) - vL(u)] dx = B(x) \bigg|_a^b.$$

What is $B(x)$? [Hint: Consider $\int_a^b vL(u) dx$ and integrate by parts.]

Appendix to 5.5: Matrix Eigenvalue Problem and Orthogonality of Eigenvectors

The matrix eigenvalue problem

$$Ax = \lambda x,$$  \hspace{1cm} (5.5.26)

where $A$ is an $n \times n$ real matrix (with entries $a_{ij}$) and $x$ is an $n$-dimensional column vector (with components $x_i$), has many properties similar to those of the Sturm-Liouville eigenvalue problem.

**Eigenvalues and eigenvectors.** For all values of $\lambda$, $x = 0$ is a "trivial" solution of the homogeneous linear system (5.5.26). We ask, for what values of $\lambda$ are there nontrivial solutions? In general, (5.5.26) can be rewritten as

$$(A - \lambda I)x = 0,$$  \hspace{1cm} (5.5.27)

where $I$ is the identity matrix. According to the theory of linear equations (elementary linear algebra), a nontrivial solution exists only if

$$\det(A - \lambda I) = 0.$$  \hspace{1cm} (5.5.28)

Such values of $\lambda$ are called eigenvalues, and the corresponding nonzero vectors $x$ called eigenvectors.

In general, (5.5.28) yields an $n$th-degree polynomial (known as the characteristic polynomial) that determines the eigenvalues; there will be $n$ eigenvalues (but they may not be distinct). Corresponding to each distinct eigenvalue, there will be an eigenvector.

**Example.** If $A = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}$, then the eigenvalues satisfy

$$0 = \det \begin{bmatrix} 2 - \lambda & 1 \\ 6 & 1 - \lambda \end{bmatrix} = (2 - \lambda)(1 - \lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1),$$

the characteristic polynomial. The eigenvalues are $\lambda = 4$ and $\lambda = -1$. For $\lambda = 4$, (5.5.26) becomes

$$2x_1 + x_2 = 4x_1 \quad \text{and} \quad 6x_1 + x_2 = 4x_2,$$

or, equivalently, $x_2 = 2x_1$. The eigenvector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an arbitrary multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda = 4$. For $\lambda = -1,$

$$2x_1 + x_2 = -x_1 \quad \text{and} \quad 6x_1 + x_2 = -x_2,$$

and thus the eigenvector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an arbitrary multiple of $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

**Green's formula.** The matrix $A$ may be thought of as a linear operator in the same way that

$$L = \frac{d}{dx} \left( p \frac{d}{dx} + q \right)$$

is a linear differential operator. $A$ operates on $n$-dimensional vectors producing an $n$-dimensional vector, while $L$ operates on functions and yields a function. In analyzing the Sturm-Liouville eigenvalue problem, Green's formula was important:

$$\int_a^b [uL(v) - vL(u)] dx = -\int_a^b p \left( \frac{du}{dx} - \frac{dv}{dx} \right) dx,$$

where $u$ and $v$ are arbitrary functions. Often, the boundary terms vanished. For vectors, the dot product is analogous to integration, $a \cdot b = \sum a_i b_i$, where $a_i$ and $b_i$ are the $i$th components of, respectively, $a$ and $b$ (see Sec. 2.3 Appendix). By direct analogy to Green's formula, we would be led to investigate $u \cdot Av$ and $v \cdot Bu$, where $u$ and $v$ are arbitrary vectors. Instead, we analyze $u \cdot Av$ and $v \cdot Bu$, where $B$ is any $n \times n$ matrix:

$$u \cdot Av = \sum_i (u_i \sum_j a_{ij} v_j) = \sum_i \sum_j a_{ij} u_i v_j$$

and $v \cdot Bu = \sum_i (v_i \sum_j b_{ij} u_j) = \sum_i \sum_j b_{ij} v_i u_j$. 

where an alternative expression for \( u \cdot Bu \) was derived by interchanging the roles of \( i \) and \( j \). Thus,

\[
u \cdot Au - v \cdot Bu = \sum_i \sum_j (a_{ij} - b_{ij}) u_i v_j.
\]

If we let \( B \) equal the transpose of \( A \) (i.e., \( b_{ij} = a_{ji} \)), whose notation is \( B = A^t \), then we have the following theorem:

\[
u \cdot Au - v \cdot A^t u = 0,
\]

(5.5.29)

analogous to Green's formula.

**Self-adjointness.** The difference between \( A \) and its transpose, \( A^t \), in (5.5.29) causes insurmountable difficulties for us. We will thus restrict our attention to symmetric matrices, in which case \( A = A^t \). For symmetric matrices

\[
u \cdot Au - v \cdot A u = 0,
\]

(5.5.30)

and we will be able to use this result to prove the same theorems about eigenvalues and eigenvectors for matrices as we proved about Sturm-Liouville eigenvalue problems.

For symmetric matrices, eigenvectors corresponding to different eigenvalues are orthogonal. To prove this, suppose that \( u \) and \( v \) are eigenvectors corresponding to \( \lambda_1 \) and \( \lambda_2 \), respectively:

\[
Au = \lambda_1 u \quad \text{and} \quad Av = \lambda_2 v.
\]

If we directly apply (5.5.30), then

\[
(\lambda_2 - \lambda_1) u \cdot v = 0.
\]

Thus, if \( \lambda_1 \neq \lambda_2 \) (different eigenvalues), the corresponding eigenvectors are orthogonal in the sense that

\[
u \cdot v = 0.
\]

(5.5.31)

We leave as an exercise the proof that the eigenvalues of a symmetric real matrix are real.

**Example.** The eigenvalues of the real symmetric matrix

\[
\begin{bmatrix}
6 & 2 \\
2 & 3
\end{bmatrix}
\]

are determined from \((6-\lambda)(3-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2) = 0. For \( \lambda = 2 \), the eigenvector satisfies

\[
6x_1 + 2x_2 = 2x_1 \quad \text{and} \quad 2x_1 + 3x_2 = 2x_2,
\]

and hence \[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= x_1 \begin{bmatrix}
1 \\
-2
\end{bmatrix}. \]

For \( \lambda = 7 \), it follows that

\[
6x_1 + 2x_2 = 7x_1 \quad \text{and} \quad 2x_1 + 3x_2 = 7x_2,
\]

and the eigenvector is \[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= x_2 \begin{bmatrix}
2 \\
1
\end{bmatrix}. \]

As we have just proved for any real symmetric matrix, the eigenvectors are orthogonal, \[
\begin{bmatrix}
1 \\
-2
\end{bmatrix} \begin{bmatrix}
2 \\
1
\end{bmatrix} = 2 - 2 = 0.
\]

**Eigenvector expansions.** For real symmetric matrices it can be shown that if an eigenvalue repeats \( R \) times, there will be \( R \) independent eigenvectors corresponding to that eigenvalue. These eigenvectors are automatically orthogonal to any eigenvectors corresponding to a different eigenvalue. The Gram-Schmidt procedure (see Sec. 6.5 Appendix) can be applied so that all \( R \) eigenvectors corresponding to the same eigenvalue can be constructed to be mutually orthogonal. In this manner, for real symmetric \( n \times n \) matrices, \( n \) orthogonal eigenvectors can always be obtained. Since these vectors are orthogonal, they span the \( n \)-dimensional vector space and may be chosen as basis vectors. Any vector \( v \) may be represented in a series of the eigenvectors:

\[
v = \sum_{i=1}^{n} c_i \phi_i.
\]

(5.5.32)

where \( \phi_i \) is the \( i \)th eigenvector. For regular Sturm-Liouville eigenvalue problems, the eigenfunctions are complete, meaning that any (piecewise smooth) function can be represented in terms of an eigenfunction expansion

\[
f(x) \sim \sum_{i=1}^{\infty} c_i \phi_i(x).
\]

(5.5.33)

This is analogous to (5.5.32). In (5.5.33) the Fourier coefficients \( c_i \) are determined by the orthogonality of the eigenfunctions. Similarly, the coordinates \( c_i \) in (5.5.32) are determined by the orthogonality of the eigenvectors. We dot equation (5.5.32) into \( \phi_m \):

\[
v \cdot \phi_m = \sum_{i=1}^{n} c_i \phi_i \cdot \phi_m = c_m \phi_m \cdot \phi_m.
\]

since \( \phi_1 \cdot \phi_m = 0, i \neq m \), determining \( c_m \).

**Linear systems.** Sturm-Liouville eigenvalue problems arise in separating variables for partial differential equations. One way in which the matrix eigenvalue problem occurs is in “separating” a linear homogeneous system of ordinary differential equations with constant coefficients. We will be very brief. A linear homogeneous first-order system of differential equations may be represented by

\[
\frac{dv}{dt} = Av,
\]

(5.5.34)
where \( A \) is an \( n \times n \) matrix and \( v \) is the desired \( n \)-dimensional vector solution. \( v \) usually satisfies given initial conditions, \( v(0) = v_0 \). We seek special solutions of the form of simple exponentials:

\[
v(t) = e^{\lambda t} \phi,
\]

where \( \phi \) is a constant vector. This is analogous to seeking product solutions by the method of separation of variables. Since \( dv/dt = \lambda e^{\lambda t} \phi \), it follows that

\[
A \phi = \lambda \phi.  \tag{5.5.36}
\]

Thus, there exist solutions to (5.5.34) of the form (5.5.35) if \( \lambda \) is an eigenvalue of \( A \) and \( \phi \) is a corresponding eigenvector. We now restrict our attention to real symmetric matrices \( A \). There will always be \( n \) mutually orthogonal eigenvectors \( \phi_i \).

We have obtained \( n \) special solutions to the linear homogeneous system (5.5.34). A principle of superposition exists, and hence a linear combination of these solutions also satisfies (5.5.34):

\[
v = \sum_{i=1}^{n} c_i e^{\lambda_i t} \phi_i. \tag{5.5.37}
\]

We attempt to determine \( c_i \) so that (5.5.37) satisfies the initial conditions, \( v(0) = v_0 \):

\[
v_0 = \sum_{i=1}^{n} c_i \phi_i.
\]

Here, the orthogonality of the eigenvectors is helpful, and thus, as before,

\[
c_i = \frac{v_0 \phi_i}{\phi_i \cdot \phi_i}.
\]

**EXERCISES 5.5 APPENDIX**

5.5A.1. Prove that the eigenvalues of real symmetric matrices are real.

5.5A.2. (a) Show that the matrix

\[
A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}
\]

has only one independent eigenvector.

(b) Show that the matrix

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

has two independent eigenvectors.

5.5A.3. Consider the eigenvectors of the matrix

\[
A = \begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}.
\]

(a) Show that the eigenvectors are not orthogonal.

(b) If the “dot product” of two vectors is defined as follows,

\[
a \cdot b = \frac{1}{2}(a_1 b_1 + a_2 b_2),
\]

show that the eigenvectors are orthogonal with this dot product.

5.5A.4. Solve \( dv/dt = Av \) using matrix methods if

* (a) \( A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}, \quad v(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}, \quad v(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \)

5.5A.5. Show that the eigenvalues are real and the eigenvectors orthogonal:

(a) \( A = \begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix} \)

* (b) \( A = \begin{bmatrix} 3 & 1 - i \\ 1 + i & 1 \end{bmatrix} \) (see Exercise 5.5A.6)

5.5A.6. For a matrix \( A \) whose entries are complex numbers, the complex conjugate of the transpose is denoted by \( A^H \). For matrices in which \( A^H = A \) (called Hermitian):

(a) Prove that the eigenvalues are real.

(b) Prove that eigenvectors corresponding to different eigenvalues are orthogonal (in the sense that \( \phi_i \cdot \phi_m = 0 \), where \( \cdot \) denotes the complex conjugate).

5.6 Rayleigh Quotient

The Rayleigh quotient can be derived from the Sturm-Liouville differential equation,

\[
\frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + q(x) \phi + \lambda \sigma(x) \phi = 0, \tag{5.6.1}
\]

by multiplying (5.6.1) by \( \phi \) and integrating:

\[
\int_a^b \left[ \phi \frac{d}{dx} \left( p \frac{d\phi}{dx} \right) + \phi \sigma \right] dx + \lambda \int_a^b \phi^2 \sigma \sigma dx = 0.
\]
Since \( \int_{a}^{b} \phi \partial^2 \sigma \, dx > 0 \), we can solve for \( \lambda \):

\[
\lambda = -\frac{\int_{a}^{b} \left[ \phi \partial \left( \frac{\partial \phi}{\partial x} \right) + q \phi^2 \right] \, dx}{\int_{a}^{b} \phi^2 \sigma \, dx}.
\]  
(5.6.2)

Integration by parts \( \int u \, dv = uv - \int v \, du \), where \( u = \phi \), \( dv = d/dx(p(\partial \phi/\partial x)) \, dx \) and hence \( du = \partial \phi/\partial x \, dx \), \( v = p \partial \phi/\partial x \) yields an expression involving the function \( \phi \) evaluated at the boundary:

\[
\lambda = -p \frac{\partial \phi}{\partial x} \bigg|_{a}^{b} + \int_{a}^{b} \left[ p \left( \frac{\partial \phi}{\partial x} \right)^2 - q \phi^2 \right] \, dx
\]  
(5.6.3)

known as the Rayleigh quotient. In Secs. 5.3 and 5.4 we have indicated some applications of this result. Further discussion will be given in Sec. 5.7.

**Nonnegative eigenvalues.** Often in physical problems, the sign of \( \lambda \) is quite important. As shown in Sec. 5.2.1, \( dh/dt + \lambda h = 0 \) in certain heat flow problems. Thus, positive \( \lambda \) corresponds to exponential decay in time, while negative \( \lambda \) corresponds to exponential growth. On the other hand, in certain vibration problems (see Sec. 5.7), \( d^2h/dt^2 = -\lambda h \). There, only positive \( \lambda \) corresponds to the "usual" expected oscillations. Thus, in both types of problems we often expect \( \lambda \geq 0 \):

The Rayleigh quotient (5.6.3) directly proves that \( \lambda \geq 0 \) if

(a) \( -p \frac{\partial \phi}{\partial x} \bigg|_{a}^{b} \geq 0 \), and

(b) \( q \leq 0 \).

We claim that both (a) and (b) are physically reasonable conditions for nonnegative \( \lambda \). Consider the boundary constraint, \( -p \frac{\partial \phi}{\partial x} \bigg|_{a}^{b} \geq 0 \). The simplest types of homogeneous boundary conditions, \( \phi = 0 \) and \( \partial \phi/\partial x = 0 \), do not contribute to this boundary term, satisfying (a). The condition \( \partial \phi/\partial x = h \phi \), in the usual cases of Newton’s law of cooling or the elastic boundary condition) has \( h > 0 \) at the left end, \( x = a \). Thus, it will have a positive contribution at \( x = a \). The sign switch at the right end, which occurs for this type of boundary condition, will also cause a positive contribution. The periodic boundary condition [e.g., \( \phi(a) = \phi(b) \) and \( p(a) \partial \phi/\partial x(a) = p(b) \partial \phi/\partial x(b) \)] as well as the singularity condition \( \phi(a) \) bounded, if \( p(a) = 0 \) also do not contribute. Thus, in all these cases \( -p \frac{\partial \phi}{\partial x} \bigg|_{a}^{b} \geq 0 \).

The source constraint \( q \leq 0 \) also has a meaning in physical problems. For heat flow problems, \( q \leq 0 \) corresponds (\( q = \alpha, Q = 0 \)) to an energy-absorbing (endothermic) reaction, while for vibration problems \( q \leq 0 \) corresponds (\( q = \alpha, Q = 0 \)) to a restoring force.

Minimization principle. The Rayleigh quotient cannot be used to determine explicitly the eigenvalue (since \( \phi \) is unknown). Nonetheless, it can be quite useful in estimating the eigenvalues. This is because of the following theorem: The minimum value of the Rayleigh quotient for all continuous functions satisfying the boundary conditions (but not necessarily the differential equation) is the lowest eigenvalue:

\[
\lambda_1 = \min \left\{ \int_{a}^{b} \left[ p \left( \frac{du}{dx} \right)^2 - qu^2 \right] \, dx \right\} / \int_{a}^{b} u^2 \sigma \, dx
\]  
(5.6.5)

where \( \lambda_1 \) represents the smallest eigenvalue. The minimization includes all continuous functions that satisfy the boundary conditions. The minimum is obtained only for \( u = \phi_1(x) \), the lowest eigenfunction. For example, the lowest eigenvalue is important in heat flow problems (see Sec. 5.4).

**Trial functions.** Before proving (5.6.5), we will indicate how (5.6.5) is applied to obtain bounds on the lowest eigenvalue. Equation (5.6.5) is difficult to apply directly since we do not know how to minimize over all functions. However, let \( u_T \) be any continuous function satisfying the boundary conditions; \( u_T \) is known as a trial function. We compute the Rayleigh quotient of this trial function, \( RQ[u_T] \):

\[
\lambda_1 \leq RQ[u_T] = -p \frac{du_T}{dx} \bigg|_{a}^{b} + \int_{a}^{b} \left[ p \left( \frac{du_T}{dx} \right)^2 - qu_T^2 \right] \, dx
\]  
(5.6.6)

We have noted that \( \lambda_1 \) must be less than or equal to the quotient since \( \lambda_1 \) is the minimum of the ratio for all functions. Equation (5.6.6) gives an upper bound for the lowest eigenvalue.

**Example.** Consider the well-known eigenvalue problem.

\[
\frac{\partial^2 \phi}{\partial x^2} + \lambda \phi = 0
\]

\[
\phi(0) = 0
\]

\[
\phi(l) = 0.
\]

We already know that \( \lambda = \pi^2 \sigma^2 (L = 1) \), and hence the lowest eigenvalue is \( \lambda_1 = \pi^2 \). For this problem, the Rayleigh quotient simplifies, and (5.6.6) becomes

\[
\lambda_1 \leq \frac{l}{l} \left( \frac{dl}{dx} \right)^2 \, dx
\]  
(5.6.7)

Trial functions must be continuous and satisfy the homogeneous boundary conditions, in this case, \( u_T(0) = 0 \) and \( u_T(1) = 0 \). In addition, we claim that the closer
the trial function is to the actual eigenfunction, the more accurate is the bound of the lowest eigenvalue. Thus, we also choose trial functions with no zeros in the interior since we already know theoretically that the lowest eigenfunction does not have a zero. We will compute the Rayleigh quotient for the three trial functions sketched in Fig. 5.6.1. For

\[ u_T = \begin{cases} \frac{x}{1-x}, & x < 0.5 \\ x^2, & x > 0.5 \end{cases} \]

(5.6.7) becomes

\[ \lambda_1 \leq \frac{\int_0^{1/2} x^2 dx + \int_{1/2}^1 (x^2 - x^2) dx}{\int_0^{1/2} x^2 dx + \int_{1/2}^1 (1-x)^2 dx} = \frac{1}{\frac{1}{2^3} + \frac{1}{2^3}} = 12, \]

a fair upper bound for the exact answer \( \pi^2(\pi^2 \approx 9.8696 \ldots) \). For \( u_T = x - x^2 \), (5.6.7) becomes

\[ \lambda_1 \leq \frac{\int_0^{1/2} (1 - 2x^2) dx + \int_{1/2}^1 (1-4x^2 + 4x^2) dx}{\int_0^{1/2} (x^2 - x^2) dx} = \frac{1 - 2 + \frac{1}{3} - \frac{1}{2} + \frac{1}{3}}{1} = 10, \]

a more accurate bound. Since \( u_T = \sin \pi x \) is the actual lowest eigenfunction, the Rayleigh quotient for this trial function will exactly equal the lowest eigenvalue. Other applications of the Rayleigh quotient will be shown in later sections.

Proof. It is usual to prove the minimization property of the Rayleigh quotient using a more advanced branch of applied mathematics known as the calculus of variations. We do not have the space here to develop that material properly. Instead, we will give a proof based on eigenfunction expansions. We again calculate the Rayleigh quotient (5.6.3) for any function \( u \) that is continuous and satisfies the homogeneous boundary conditions. In this derivation, the equivalent form of the Rayleigh quotient, (5.6.2), is more useful:

\[ RQ[u] = -\int_0^L u L(u) \, dx \int_0^L u^2 \sigma \, dx, \]

where the operator notation is quite helpful. We expand the rather arbitrary function \( u \) in terms of the (usually unknown) eigenfunctions \( \phi_n(x) \):

\[ u = \sum_{n=1}^{\infty} a_n \phi_n(x). \]

(5.6.9)

\( L \) is a linear differential operator. We expect that

\[ L(u) = \sum_{n=1}^{\infty} a_n L(\phi_n(x)), \]

(5.6.10)

since this is valid for finite series. In Chapter 7 we show that (5.6.10) is valid if \( u \) is continuous and satisfies the same homogeneous boundary conditions as the eigenfunctions \( \phi_n(x) \). Here, \( \phi_n \) are eigenfunctions, and hence \( L(\phi_n) = -\lambda_n \phi_n \).

Thus, (5.6.10) becomes

\[ L(u) = -\sum_{n=1}^{\infty} a_n \lambda_n \phi_n, \]

(5.6.11)

which can be thought of as the eigenfunction expansion of \( L(u) \). If (5.6.11) and (5.6.9) are substituted into (5.6.8) and different dummy summation indices are utilized for the product of two infinite series, we obtain

\[ RQ[u] = \int_0^L \left( \sum_{n=1}^{\infty} a_n \phi_n \right) \left( \sum_{m=1}^{\infty} a_m \phi_m \right) \, dx \int_0^L \left( \sum_{n=1}^{\infty} a_n \phi_n \right) \left( \sum_{m=1}^{\infty} a_m \phi_m \right) \, dx. \]

We now do the integration in (5.6.12) before the summation. We recall that the eigenfunctions are orthogonal

\[ \int_0^L \phi_n \phi_m \, dx = 0 \quad \text{if} \quad n \neq m, \]

which implies that (5.6.12) becomes

\[ RQ[u] = \sum_{n=1}^{\infty} a_n^2 \lambda_n = \sum_{n=1}^{\infty} \int_0^L \phi_n^2 \sigma \, dx. \]

(5.6.13)

This is an exact expression for the Rayleigh quotient in terms of the generalized Fourier coefficients \( a_n \) of \( u \). We denote \( \lambda_1 \) as the lowest eigenvalue \( (\lambda_1 < \lambda_n \text{ for } n > 1) \). Thus,

\[ RQ[u] \geq \lambda_1 \int_0^L \phi_1^2 \sigma \, dx = \lambda_1. \]

(5.6.14)

Furthermore, the equality in (5.6.14) holds only if \( a_n = 0 \) for \( n > 1 \) (i.e., only if \( u = a_1 \phi_1 \)). We have shown that the smallest value of the Rayleigh quotient is the lowest eigenvalue \( \lambda_1 \). Moreover, the Rayleigh quotient is minimized only when \( u = a_1 \phi_1 \) (i.e., when \( u \) is the lowest eigenfunction).
We thus have a minimization theorem for the lowest eigenvalue $\lambda_1$. We can ask if there are corresponding theorems for the higher eigenvalues. Interesting generalizations immediately follow from (5.6.13). If we insist that $a_1 = 0$, then

$$RQ[u] = \sum_{n=2}^{\infty} a_n^2 \lambda_n \int_0^1 \phi_n^2(x) \, dx. \quad (5.6.15)$$

This means that in addition we are restricting our function $u$ to be orthogonal to $\phi_1$, since $a_1 = 0$. Hence, $RQ[u] \geq \lambda_2$, and furthermore the equality holds only if $a_n = 0$ for $n > 2$ [i.e., $u = a_2 \phi_2(x)$] since $a_1 = 0$ already. We have just proved the following theorem: The minimum value for all continuous functions $u(x)$ that are orthogonal to the lowest eigenfunction and satisfy the boundary conditions is the next-to-lowest eigenvalue. Further generalizations also follow directly from (5.6.13).

**EXERCISES 5.6**

5.6.1. Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of

(a) $\frac{d^2 \phi}{dx^2} + (\lambda - x^2) \phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\phi(1) = 0$

(b) $\frac{d^2 \phi}{dx^2} + (\lambda - x) \phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) + 2\phi(1) = 0$

(c) $\frac{d^2 \psi}{dx^2} + \lambda \psi = 0$ with $\psi(0) = 0$ and $\frac{d\psi}{dx}(1) + \phi(1) = 0$ (See Exercise 5.8.10.)

5.6.2. Consider the eigenvalue problem

$$\frac{d^2 \phi}{dx^2} + (\lambda - x^2) \phi = 0$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).

5.6.3. Prove that (5.6.10) is valid in the following way. Assume $L(u)/\sigma$ is piecewise smooth so that

$$\frac{L(u)}{\sigma} = \sum_{n=1}^{\infty} b_n \phi_n(x).$$

Determine $b_n$. [Hint: Using Green’s formula (5.5.5), show that $b_n = -\alpha_n \lambda_n$ if $u$ and $du/dx$ are continuous and if $u$ satisfies the same homogeneous boundary conditions as the eigenfunctions $\phi_n(x)$.]
guarantees that \( \lambda > 0 \). For (5.7.6), the Rayleigh quotient (5.6.3) becomes

\[
\lambda = \frac{T_0 \int_{0}^{L} \left( \frac{d}{dx} \phi \right)^2 dx}{\int_{0}^{L} \phi^2 p(x) dx}.
\]

(5.7.7)

Clearly, \( \lambda \geq 0 \) (and as before it is impossible for \( \lambda = 0 \) in this case). Thus, \( \lambda > 0 \).

We now are assured that the solution of (5.7.5) is a linear combination of \( \sin \sqrt{\lambda_n t} \phi_n(x) \) and \( \cos \sqrt{\lambda_n t} \phi_n(x) \). According to the principle of superposition, the solution is

\[
u(x, t) = \sum_{n=1}^{\infty} a_n \sin \sqrt{\lambda_n t} \phi_n(x) + \sum_{n=1}^{\infty} b_n \cos \sqrt{\lambda_n t} \phi_n(x).
\]

(5.7.8)

We only need to show that the two families of coefficients can be obtained from the initial conditions:

\[
f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} a_n \sqrt{\lambda_n} \phi_n(x).
\]

(5.7.9)

Thus, \( b_n \) are the generalized Fourier coefficient of the initial position \( f(x) \) while \( a_n \sqrt{\lambda_n} \) are the generalized Fourier coefficients for the initial velocity \( g(x) \). Thus, due to the orthogonality of the eigenfunction [with weight \( p(x) \)], we can easily determine \( a_n \) and \( b_n \):

\[
b_n = \frac{\int_{0}^{L} f(x) \phi_n(x) p(x) dx}{\int_{0}^{L} \phi_n^2 p(x) dx} \quad (5.7.10)
\]

\[
a_n \sqrt{\lambda_n} = \frac{\int_{0}^{L} g(x) \phi_n(x) p(x) dx}{\int_{0}^{L} \phi_n^2 p(x) dx} \quad (5.7.11)
\]

The Rayleigh quotient can be used to obtain additional information about the lowest eigenvalue \( \lambda_1 \). (Note that the lowest frequency of vibration is \( \sqrt{\lambda_1} \).) We know that

\[
\lambda_1 = \min \frac{T_0 \int_{0}^{L} \left( \frac{d}{dx} u \right)^2 dx}{\int_{0}^{L} u^2 p(x) dx}.
\]

(5.7.12)

We have already shown (see Sec. 5.6) how to use trial functions to obtain an upper bound on the lowest eigenvalue. This is not always convenient since the denominator in (5.7.12) depends on the mass density \( p(x) \). Instead, we will develop another method for an upper bound. By this method we will also obtain a lower bound.

Let us suppose, as is usual, that the variable mass density has upper and lower bounds,

\[0 < \rho_{\text{min}} \leq \rho(x) \leq \rho_{\text{max}}.<\]

For any \( u(x) \) it follows that

\[
\rho_{\text{min}} \int_{0}^{L} u^2 dx \leq \int_{0}^{L} u^2 p(x) dx \leq \rho_{\text{max}} \int_{0}^{L} u^2 dx.
\]

Consequently, from (5.7.12),

\[
\frac{T_0}{\rho_{\text{max}}} \left( \int_{0}^{L} \left( \frac{d}{dx} u \right)^2 dx \right) \leq \lambda_1 \leq \frac{T_0}{\rho_{\text{min}}} \left( \int_{0}^{L} \left( \frac{d}{dx} u \right)^2 dx \right).
\]

(5.7.13)

We can evaluate the expressions in (5.7.13), since we recognize the minimum of \( \int_{0}^{L} \left( \frac{d}{dx} u \right)^2 dx / \int_{0}^{L} u^2 dx \) subject to \( u(0) = 0 \) and \( u(L) = 0 \) as the lowest eigenvalue of a different problem: namely, one with constant coefficients,

\[
d \frac{d^2 \phi}{dx^2} + \lambda \phi = 0
\]

\[
\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.
\]

We already know that \( \lambda_1 = \left( \pi / L \right)^2 \), and hence the lowest eigenvalue for this problem is \( \lambda_1 = \left( \pi / L \right)^2 \). But the minimization property of the Rayleigh quotient implies that

\[
\lambda_1 = \min \frac{T_0}{\rho_{\text{max}}} \left( \frac{\pi}{L} \right)^2 \lambda_1 \leq \frac{T_0}{\rho_{\text{min}}} \left( \frac{\pi}{L} \right)^2.
\]

Finally, we have proved that the lowest eigenvalue of our problem with variable coefficients satisfies the following inequality:

\[
\frac{T_0}{\rho_{\text{max}}} \left( \frac{\pi}{L} \right)^2 \leq \lambda \leq \frac{T_0}{\rho_{\text{min}}} \left( \frac{\pi}{L} \right)^2.
\]

We have obtained an upper and a lower bound for the smallest eigenvalue. By taking square roots,

\[
\pi \sqrt{\frac{T_0}{\rho_{\text{max}}}} \leq \sqrt{\lambda_1} \leq \pi \sqrt{\frac{T_0}{\rho_{\text{min}}}}.
\]

The physical meaning of this is clear: the lowest frequency of oscillation of a variable string lies in between the lowest frequencies of vibration of two constant density strings, one with the minimum density and the other with the maximum. Similar results concerning the higher frequencies of vibration are also valid but are harder to prove (see Weinberger [1965] or Courant and Hilbert [1953]).
EXERCISES 5.7

5.7.1. Determine an upper and a (nonzero) lower bound for the lowest frequency of vibration of a nonuniform string fixed at $x = 0$ and $x = 1$ with $c^2 = 1 + 4a^2(x - \frac{1}{2})^2$.

5.7.2. Consider heat flow in a one-dimensional rod without sources with nonconstant thermal properties. Assume that the temperature is zero at $x = 0$ and $x = L$.

**Physical examples.** We consider some simple problems with constant physical parameters. Heat flow in a uniform rod satisfies

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

while a uniform vibrating string solves

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

In either case we suppose that the left end is fixed, but the right end satisfies a homogeneous boundary condition of the third kind:

$$u(0, t) = 0$$

$$\frac{\partial u}{\partial x}(L, t) = -hu(L, t).$$

Recall that, for heat conduction, (5.8.4) corresponds to Newton’s law of cooling if $h > 0$, and for the vibrating string problem, (5.8.4) corresponds to a restoring force if $h > 0$, the so-called elastic boundary condition. We note that usually in physical problems $h > 0$. However, for mathematical reasons we will investigate both cases with $h < 0$ and $h > 0$. If $h < 0$, the vibrating string has a destabilizing force at the right end, while for the heat flow problem, thermal energy is being constantly put into the rod through the right end.

**Sturm-Liouville eigenvalue problem.** After separation of variables, $u(x, t) = G(t)\phi(x)$, the time part satisfies the following ordinary differential equations:

heat flow:

$$\frac{dG}{dt} = -\lambda k G$$

vibrating string:

$$\frac{d^2 G}{dt^2} = -\lambda c^2 G.$$
Clearly, sine functions are needed to satisfy the zero condition at \( x = 0 \). We will need the first derivative, 
\[
\frac{d\phi}{dx} = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x.
\]
Thus, the boundary condition of the third kind, (5.8.10), implies that
\[
c_2 (\sqrt{\lambda} \cos \sqrt{\lambda} L + h \sin \sqrt{\lambda} L) = 0.
\]  
(5.8.13)

If \( c_2 = 0 \), (5.8.12) shows that \( \phi \equiv 0 \), which cannot be an eigenfunction. Thus, eigenvalues exist for \( \lambda > 0 \) for all values of \( \lambda \) that satisfy
\[
\sqrt{\lambda} \cos \sqrt{\lambda} L + h \sin \sqrt{\lambda} L = 0.
\]  
(5.8.14)

The more elementary case \( h = 0 \) will be analyzed later. Equation (5.8.14) is a transcendental equation for the positive eigenvalues \( \lambda \) (if \( h \neq 0 \)). In order to solve (5.8.14), it is convenient to divide by \( \cos \sqrt{\lambda} L \) to obtain an expression for \( \tan \sqrt{\lambda} L \):
\[
\tan \sqrt{\lambda} L = \frac{-\sqrt{\lambda}}{h}.
\]  
(5.8.15)

We are allowed to divide by \( \cos \sqrt{\lambda} L \) because it is not zero [if \( \cos \sqrt{\lambda} L = 0 \), then \( \sin \sqrt{\lambda} L \neq 0 \) and (5.8.14) would not be satisfied]. We could have obtained an expression for cotangent rather than tangent by dividing (5.8.14) by \( \sin \sqrt{\lambda} L \), but we are presuming that the reader feels more comfortable with the tangent function.

**Graphical technique (\( \lambda > 0 \)).** Equation (5.8.15) is a transcendental equation. We cannot solve it exactly. However, let us describe a graphical technique to obtain information about the eigenvalues. In order to graph the solution of a transcendental equation, we introduce an artificial coordinate \( z \). Let
\[
z = \tan \sqrt{\lambda} L
\]  
(5.8.16)

and thus also
\[
z = \frac{-\sqrt{\lambda}}{h}.
\]  
(5.8.17)

Now the simultaneous solution of (5.8.16) and (5.8.17) (i.e., their points of intersection) corresponds to solutions of (5.8.15). Equation (5.8.16) is a pure tangent function (not compressed) as a function of \( \sqrt{\lambda} L \), where \( \sqrt{\lambda} L > 0 \) since \( \lambda > 0 \). We sketch (5.8.16) in Fig. 5.8.1. We note that the tangent function is periodic with period \( \pi \); it is zero at \( \sqrt{\lambda} L = 0, \pi, 2\pi \), and etc.; and it approaches \( \pm \infty \) as \( \sqrt{\lambda} L \) approaches \( \pi/2, 3\pi/2, 5\pi/2 \), and so on. We will intersect the tangent function with (5.8.17). Since we are sketching our curves as functions of \( \sqrt{\lambda} L \), we will express (5.8.17) as a function of \( \sqrt{\lambda} L \). This is easily done by multiplying numerator and denominator of (5.8.17) by \( L \):

\[
z = -\frac{\sqrt{\lambda} L}{hL}.
\]  
(5.8.18)

As a function of \( \sqrt{\lambda} L \), (5.8.18) is a straight line with slope \(-1/hL\). However, this line is sketched quite differently depending on whether \( h > 0 \) (physical case) or \( h < 0 \) (nonphysical case).

**Positive eigenvalues (physical case, \( h > 0 \)).** The intersection of the two curves is sketched in Fig. 5.8.1 for the physical case (\( h > 0 \)). There is an infinite number of intersections; each corresponds to a positive eigenvalue. (We exclude \( v' \L = 0 \) since we have assumed throughout that \( \lambda > 0 \).) The eigenfunctions are \( \phi = \sin \sqrt{\lambda} x \), where the allowable eigenvalues are determined graphically.

![Figure 5.8.1](image_url)

**Figure 5.8.1** Graphical determination of positive eigenvalues (\( h > 0 \)).

We cannot determine these eigenvalues exactly. However, we know from Fig. 5.8.1 that
\[
\frac{\pi}{2} < \sqrt{\lambda_1} L < \pi,
\]  
(5.8.19)

\[
\frac{3\pi}{2} < \sqrt{\lambda_2} L < 2\pi,
\]  
(5.8.20)

and so on. It is interesting to note that as \( n \) increases, the intersecting points move more closely approach the position of the vertical portions of the tangent function. We thus are able to obtain the following approximate (asymptotic) formula for the eigenvalues
\[
\sqrt{\lambda_n} L \sim (n - \frac{1}{2})\pi
\]  
(5.8.21)

as \( n \to \infty \). This becomes more and more accurate as \( n \to \infty \). An asymptotic formula for the large eigenvalues similar to (5.8.21) exists even for cases where the differential equation cannot be explicitly solved. We will discuss this in Sec. 5.9.
To obtain accurate values, a numerical method such as Newton's method (as often described in elementary calculus texts) can be used. A practical scheme is to use Newton's numerical method for the first few roots, until you reach a root whose solution is reasonably close to the asymptotic formula, (5.8.21) (or improvements to this elementary asymptotic formula). Then, for larger roots, the asymptotic formula (5.8.21) is accurate enough.

**Positive eigenvalues (nonphysical case, $h < 0$).** The nonphysical case ($h < 0$) also will be a good illustration of various general ideas concerning Sturm-Liouville eigenvalue problems. If $h < 0$, positive eigenvalues again are determined by graphically sketching (5.8.15), $\tan \sqrt{hL} = -\sqrt{h}/L$. The straight line (here with positive slope, must intersect the tangent function. It intersects the “first branch” of the tangent function only if the slope of the straight line is greater than 1 (see Fig. 5.8.2a). We are using the property of the tangent function that its slope is 1 at $x = 0$ and its slope increases along the first branch. Thus, if $h < 0$ (the nonphysical case), there are two major subcases ($-1/hL > 1$ and $0 < -1/hL < 1$) and a minor subcase ($-1/hL = 1$). We sketch these three cases in Fig. 5.8.2. In each of these three figures, there is an infinite number of intersections, corresponding to an infinite number of positive eigenvalues. The eigenfunctions are again $\sin \sqrt{h}x$.

![Graphical determination of positive eigenvalues](image)

In these cases, the graphical solutions also show that the large eigenvalues are approximately located at the singularities of the tangent function. Equation (5.8.21) is again asymptotic; the larger is $n$, the more accurate is (5.8.21).

**Zero eigenvalue.** Is $\lambda = 0$ an eigenvalue for (5.8.8)-(5.8.10)? Equation (5.8.11) is not the general solution of (5.8.8) if $\lambda = 0$. Instead,

$$\phi = c_1 + c_2 x;$$

(5.8.22)

the eigenfunction must be a straight line. The boundary condition $\phi(0) = 0$ makes $c_1 = 0$, insisting that the straight line goes through the origin,

$$\phi = c_2 x.$$  

(5.8.23)

Finally, $d\phi/dx(L) + h\phi(L) = 0$ implies that

$$c_2(1 + hL) = 0.$$  

(5.8.24)

If $hL \neq -1$ (including all physical situations, $h > 0$), it follows that $c_2 = 0$, $\phi = 0$, and thus $\lambda = 0$ is not an eigenvalue. However, if $hL = -1$, then from (5.8.24) $c_2$ is arbitrary, and $\lambda = 0$ is an eigenvalue with eigenfunction $x$.

**Negative eigenvalues.** We do not expect any negative eigenvalues in the physical situations [see (5.8.6) and (5.8.7)]. If $\lambda < 0$ we introduce $s = -\lambda$, so that $s > 0$. Then (5.8.8) becomes

$$\frac{d^2\phi}{dx^2} = -s \phi.$$  

(5.8.25)

The zero boundary condition at $x = 0$ suggests that it is more convenient to express the general solution of (5.8.25) in terms of the hyperbolic functions:

$$\phi = c_1 \cosh \sqrt{s}x + c_2 \sinh \sqrt{s}x.$$  

(5.8.26)

Only the hyperbolic sines are needed, since $\phi(0) = 0$ implies that $c_1 = 0$:

$$\frac{d\phi}{dx} = c_2 \sqrt{s} \cosh \sqrt{s}x.$$  

(5.8.27)

The boundary condition of the third kind, $d\phi/dx(L) + h\phi(L) = 0$, implies that

$$c_2(\sqrt{s} \cosh \sqrt{s}L + h \sinh \sqrt{s}L) = 0.$$  

(5.8.28)

At this point it is apparent that the analysis for $\lambda < 0$ directly parallels that which occurred for $\lambda > 0$ (with hyperbolic functions replacing the trigonometric functions). Thus, since $c_2 \neq 0$,

$$\tanh \sqrt{s}L = -\frac{sL}{h} = -\sqrt{hL}.$$  

(5.8.29)

**Graphical solution for negative eigenvalues.** Negative eigenvalues are determined by the graphical solution of transcendental equation (5.8.29). Here properties of the hyperbolic tangent function are quite important. tanh is sketched as a function of $\sqrt{s}L$ in Fig. 5.8.3. Let us note some properties of the tanh function that follow from its definition:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$
As $\sqrt{sL} \to \infty$, $\tanh \sqrt{sL}$ asymptotes to 1. We will also need to note that the slope of $\tanh x$ equals 1 at $\sqrt{sL} = 0$ and decreases toward zero as $\sqrt{sL} \to \infty$. This function must be intersected with the straight line implied by the r.h.s. of (5.8.29). The same four cases appear, as is sketched in Fig. 5.8.3. In physical situations ($h > 0$), there are no intersections with $\sqrt{sL} > 0$; there are no negative eigenvalues in the physical situations ($h > 0$). All the eigenvalues are nonnegative. However, if $hL < -1$ (and only in these situations), then there is exactly one intersection; there is one negative eigenvalue (if $hL < -1$). If we denote the intersection by $\theta = \theta_1$ the negative eigenvalue is $\lambda = -\theta_1$, and the corresponding eigenfunction is $\psi = \sinh \sqrt{\theta_1}x$. In nonphysical situations, there is a finite number of negative eigenvalues (one if $hL < -1$, none otherwise).

Special case $h = 0$. Although if $h = 0$, the boundary conditions are not of the third kind, the eigenvalues and eigenfunctions are still of interest. If $h = 0$, then all eigenvalues are positive [see (5.8.24) and (5.8.28)] and easily explicitly determined from (5.8.14):

$$\lambda = \left(\frac{n - 1/2}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

The eigenfunctions are $\sin \sqrt{\lambda}x$.

Summary. We have shown there to be five somewhat different cases depending on the value of the parameter $h$ in the boundary condition. Table 5.8.1 summarizes the eigenvalues and eigenfunctions for these cases.

In some sense there actually are only three cases: If $-1 < hL$, all the eigenvalues are positive; if $hL = -1$, there are no negative eigenvalues, but zero is an eigenvalue; and if $hL < -1$, there are still an infinite number of positive eigenvalues, but there is also one negative one.

Rayleigh quotient. We have shown by explicitly solving the eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda \phi = 0 \quad (5.8.30)$$
$$\phi(0) = 0 \quad (5.8.31)$$
$$h\phi(L) = 0, \quad (5.8.32)$$

that in physical problems ($h > 0$) all the eigenvalues are positive, while in nonphysical problems ($h < 0$) there may or may not be negative eigenvalues. We will show that the Rayleigh quotient is consistent with this result:

$$\lambda = \frac{-p\phi(0)^2 + \int_a^b \left[ p \left( \frac{d\phi}{dx} \right)^2 - q\phi^2 \right] dx}{\int_a^b \phi^2 dx} \quad (5.8.33)$$

since from (5.8.30), $p(x) = 1, q(x) = 1, q(x) = 0$, and $a = 0, b = L$, and where the boundary conditions (5.8.31) and (5.8.32) have been utilized to simplify the boundary terms in the Rayleigh quotient. If $h \geq 0$ (the physical cases), it readily follows from (5.8.33) that the eigenvalues must be positive, exactly what we concluded by doing the explicit calculations. However, if $h < 0$ (nonphysical case), the numerator of the Rayleigh quotient contains a negative term $h\phi^2(L)$ and a positive term $\int_a^b \left( \frac{d\phi}{dx} \right)^2 dx$. It is impossible to make any conclusions concerning the sign.

| Physical | $h > 0$ | $\sin \sqrt{\lambda}x$ |
| Nonphysical | $hL = -1$ | $\sin \sqrt{\lambda}x$ | $x$ |
| $hL < -1$ | $\sin \sqrt{\lambda}x$ | $\sinh \sqrt{\lambda}x$ |
of \lambda. Thus, it may be possible to have negative eigenvalues if \( h < 0 \). However, we are unable to conclude that there must be negative eigenvalues. A negative eigenvalue occurs only when \( |h\phi''(L)| > \int_0^L (d\phi/dx)^2 \, dx \).

From the Rayleigh quotient we cannot determine when this happens. It is only from an explicit calculation that we know that a negative eigenvalue occurs only if \( hL < -1 \).

Zeros of eigenfunctions. The Sturm-Liouville eigenvalue problem that we have been discussing in this section,

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0, \]

is a good example for illustrating the general theorem concerning the zeros of the eigenfunctions. The theorem states that the eigenfunction corresponding to the lowest eigenvalue has no zeros in the interior. More generally, the \( n \)th eigenfunction has \( n-1 \) zeros.

There are five cases of (5.8.34) worthy of discussion: \( h > 0, h = 0, -1 < hL < 0, hL = -1, hL < -1 \). However, the line of reasoning used in investigating the zeros of the eigenfunctions is quite similar in all cases. For that reason we will analyze only one case \( (hL < -1) \) and leave the others for the exercises. In this case \( (hL < -1) \) there is one negative eigenvalue (with corresponding eigenfunction \( \sinh \sqrt{hL} x \)) and an infinite number of positive eigenvalues (with corresponding eigenfunctions \( \sin \sqrt{\lambda} x \)). We will need to analyze carefully the positive eigenvalues and so we reproduce Fig. 5.8.4 (as Fig. 5.8.5), used for the graphical determination of the eigenvalues in \( hL < -1 \). We designate the intersections starting from \( \lambda_n, n = 2 \), since the lowest eigenvalue is negative, \( \lambda_1 = -\lambda_1 \). Graphically, we are able to obtain bounds for these eigenvalues:

\[ \pi < \sqrt{\lambda_2} L < \frac{3\pi}{2} \]
\[ 2\pi < \sqrt{\lambda_3} L < \frac{5\pi}{2} \].

Let us investigate zeros of the eigenfunctions. The lowest eigenfunction is \( \sinh \sqrt{\lambda_1} x \). Since the hyperbolic sine function is never zero (except at the end \( x = 0 \)), we have verified one part of the theorem. The eigenfunction corresponding to the lowest eigenvalue does not have a zero in the interior. The other eigenfunctions are \( \sin \sqrt{\lambda_2} x \), sketched in Fig. 5.8.5. In this figure the endpoint \( x = 0 \) is clearly marked, but \( x = L \) depends on \( \lambda \). For example, for \( \lambda_3 \), the endpoint \( x = L \) occurs at \( \sqrt{\lambda_3} L \), which is sketched in Fig. 5.8.5 due to (5.8.36). As \( x \) varies from 0 to \( L \), the eigenfunction is sketched in Fig. 5.8.5 up to the dashed line. This eigenfunction has two zeros \( (\sqrt{\lambda_2} \pi = \pi \) and \( 2\pi \)). This reasoning can be used for any of these eigenfunctions. Thus, the number of zeros for the \( n \)th eigenfunction corresponding to \( \lambda_n \), is \( n-1 \), exactly as the general theorem specifies. Our theorem does not state that the eigenfunction corresponding to the lowest positive eigenvalue has no zeros. Instead, the eigenfunction corresponding to the lowest eigenvalue has no zeros. To repeat, in this example the lowest eigenvalue is negative and its corresponding eigenfunction has no zeros.

Heat flow with a nonphysical boundary condition. To understand further the boundary condition of the third kind, let us complete the investigation of one example. We consider heat flow in a uniform rod:

\[ \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \]

BC1: \( u(0, t) = 0 \)

BC2: \( \frac{\partial u}{\partial x}(L, t) = -hu(L, t) \)

IC: \( u(x, 0) = f(x) \)

We assume that the temperature is zero at \( x = 0 \), and that the "nonphysical" case \( (h < 0) \) of the boundary condition of the third kind is imposed at \( x = L \). Thermal energy flows into the rod at \( x = L \) [if \( u(L, t) > 0 \)]. Separating variables,

\[ u(x, t) = \phi(x)G(t) \]

which is easily generalized as

\[ (n-1)\pi < \sqrt{\lambda_n} L < (n-1/2)\pi \quad n \geq 2. \]

Let us investigate zeros of the eigenfunctions. The lowest eigenfunction is \( \sinh \sqrt{\lambda_1} x \). Since the hyperbolic sine function is never zero (except at the end \( x = 0 \)), we have verified one part of the theorem. The eigenfunction corresponding to the lowest eigenvalue does not have a zero in the interior. The other eigenfunctions are \( \sin \sqrt{\lambda_2} x \), sketched in Fig. 5.8.5. In this figure the endpoint \( x = 0 \) is clearly marked, but \( x = L \) depends on \( \lambda \). For example, for \( \lambda_3 \), the endpoint \( x = L \) occurs at \( \sqrt{\lambda_3} L \), which is sketched in Fig. 5.8.5 due to (5.8.36). As \( x \) varies from 0 to \( L \), the eigenfunction is sketched in Fig. 5.8.5 up to the dashed line. This eigenfunction has two zeros \( (\sqrt{\lambda_2} \pi = \pi \) and \( 2\pi \)). This reasoning can be used for any of these eigenfunctions. Thus, the number of zeros for the \( n \)th eigenfunction corresponding to \( \lambda_n \), is \( n-1 \), exactly as the general theorem specifies. Our theorem does not state that the eigenfunction corresponding to the lowest positive eigenvalue has no zeros. Instead, the eigenfunction corresponding to the lowest eigenvalue has no zeros. To repeat, in this example the lowest eigenvalue is negative and its corresponding eigenfunction has no zeros.
yields
\[ \frac{dG}{dt} = -\lambda k G \]  
(5.8.40)
\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \]  
(5.8.41)
\[ \phi(0) = 0 \]  
(5.8.42)
\[ \frac{d\phi}{dx}(L) + h \phi(L) = 0. \]  
(5.8.43)

The time part is an exponential, \( G = e^{-\lambda k t} \). Here, we only consider the case in which

\[ hL < -1. \]

Then there exists one negative eigenvalue (\( \lambda_L = -s_1 \)), with corresponding eigenfunction \( \sinh \sqrt{s_1} x \), where \( s_1 \) is determined as the unique solution of \( \tanh \sqrt{s_1} = -\sqrt{s_1}/h \). The time part exponentially grows. All the other eigenvalues \( \lambda_n \) are positive. For these the eigenfunctions are \( \sin \sqrt{\lambda_n} x \) (where \( \tan \sqrt{\lambda_n} = -\sqrt{\lambda_n}/h \) has an infinite number of solutions), while the corresponding time-dependent part exponentially decays being proportional to \( e^{-\lambda_n k t} \). The forms of the product solutions \( S_k t \) are \( \sin \sqrt{\lambda_n} x e^{-\lambda_n k t} \) and \( \sinh \sqrt{s_1} x e^{-s_1 k t} \). Here, the somewhat “abstract” notation may be considered more convenient; the product solutions are

\[ \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n k t}, \]

where the eigenfunctions are

\[ \phi_n(x) = \begin{cases} \sinh \sqrt{s_1} x & n = 1 \\ \sin \sqrt{\lambda_n} x & n > 1. \end{cases} \]

According to the principle of superposition, we attempt to satisfy the initial value problem with a linear combination of all possible production solutions:

\[ u(x, t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n k t}. \]

The initial condition, \( u(x, 0) = f(x) \), implies that

\[ f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x). \]

Since the coefficient \( \sigma(x) = 1 \) in (5.8.41), the eigenfunctions \( \phi_n(x) \) are orthogonal with weight 1. Thus, we know that the generalized Fourier coefficients of the initial condition \( f(x) \) are

\[ a_n = \int_0^L f(x) \phi_n(x) dx \]

subject to \( a_n = \int_0^L f(x) \phi_n(x) dx \)

(5.8.41)

(5.8.44)

In particular, we could show \( \int_0^L \sin^2 \sqrt{\lambda_n} x \sin \lambda_n x dx = 0(n \neq m) \) and \( \int_0^L \sin \sqrt{s_1} x \sin \sqrt{s_1} x dx = 0 \). We do not need to verify these by integration (although it can be done).

Other problems with boundary conditions of the third kind appear in the Exercises.

EXERCISES 5.8

5.8.1. Consider

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]

subject to \( u(0, t) = 0, \frac{\partial u}{\partial x}(L, t) = -hu(L, t) \), and \( u(x, 0) = f(x) \).

(a) Solve if \( hL > -1 \).

(b) Solve if \( hL = -1 \).

5.8.2. Consider the eigenvalue problem (5.8.8)-(5.8.10). Show that the nth eigenfunction has \( n - 1 \) zeros in the interior if

(a) \( h > 0 \)

(b) \( h = 0 \)

(c) \( -1 < hL < 0 \)

(d) \( hL = -1 \)

5.8.3. Consider the eigenvalue problem

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0, \]

subject to \( \frac{d\phi}{dx}(0) = 0 \) and \( \frac{d\phi}{dx}(L) + h \phi(L) = 0 \) with \( h > 0 \).

(a) Prove that \( \lambda > 0 \) (without solving the differential equation).

(b) Determine all eigenvalues graphically. Obtain upper and lower bounds. Estimate the large eigenvalues.

(c) Show that the nth eigenfunction has \( n - 1 \) zeros in the interior.

5.8.4. Redo Exercise 5.8.3 parts (b) and (c) only if \( h < 0 \).
5.8.5. Consider
\[ \frac{\partial^2 u}{\partial t^2} = k \frac{\partial^2 u}{\partial x^2} \]
with \( \frac{\partial u}{\partial t}(0, t) = 0 \), \( \frac{\partial u}{\partial t}(L, t) = -hu(L, t) \), and \( u(x, 0) = f(x) \).
(a) Solve if \( h > 0 \).
(b) Solve if \( h < 0 \).

5.8.6. Consider (with \( h > 0 \))
\[ \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \]
\[ \frac{\partial u}{\partial t}(0, t) - hu(0, t) = 0 \quad u(x, 0) = f(x) \]
\[ \frac{\partial u}{\partial x}(L, t) = 0 \quad \frac{\partial u}{\partial t}(x, 0) = g(x). \]
(a) Show that there are an infinite number of different frequencies of oscillation.
(b) Estimate the large frequencies of oscillation.
(c) Solve the initial value problem.

*5.8.7. Consider the eigenvalue problem
\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \]
subject to \( \phi(0) = 0 \) and \( \phi(x) - 2 \frac{d\phi}{dx}(0) = 0 \).
(a) Show that usually
\[ \int_0^x \left( u \frac{d^2 v}{dx^2} - v \frac{d^2 u}{dx^2} \right) \, dx \neq 0 \]
for any two functions \( u \) and \( v \) satisfying these homogeneous boundary conditions.
(b) Determine all positive eigenvalues.
(c) Determine all negative eigenvalues.
(d) Is \( \lambda = 0 \) an eigenvalue?
(e) Is it possible that there are other eigenvalues besides those determined in parts (b) through (d)? Briefly explain.

5.8.8. Consider the boundary value problem
\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \quad \phi(0) = \frac{d\phi}{dx}(0) = 0 \]
\[ \phi(1) + \frac{d\phi}{dx}(1) = 0. \]
(a) Using the Rayleigh quotient, show that \( \lambda \geq 0 \). Why is \( \lambda > 0 \)?

(b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.
* (c) Show that
\[ \tan \sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1} \]
Determine the eigenvalues graphically. Estimate the large eigenvalues.
(d) Solve
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]
with
\[ u(0, t) - \frac{\partial u}{\partial x}(0, t) = 0 \]
\[ u(1, t) + \frac{\partial u}{\partial x}(1, t) = 0 \]
\[ u(x, 0) = f(x). \]
You may call the relevant eigenfunctions \( \phi_n(x) \) and assume that they are known.

5.8.9. Consider the eigenvalue problem
\[ \frac{d^2 \phi}{dx^2} + \beta \phi = 0 \quad \phi(0) = \frac{d\phi}{dx}(0) \quad \phi(1) = \beta \frac{d\phi}{dx}(1). \]
For what values (if any) of \( \beta \) is \( \lambda = 0 \) an eigenvalue?

5.8.10. Consider the special case of the eigenvalue problem of Sec. 5.8:
\[ \frac{d^2 \phi}{dx^2} + \beta \phi = 0 \quad \frac{d\phi}{dx}(0) + \phi(1) = 0. \]
* (a) Determine the lowest eigenvalue to at least two or three significant figures using tables or a calculator.
* (b) Determine the lowest eigenvalue using a root finding algorithm (e.g., Newton’s method) on a computer.
(c) Compare either part (a) or (b) to the bound obtained using the Rayleigh quotient [see Exercise 5.6.1(c)].

5.8.11. Determine all negative eigenvalues for
\[ \frac{d^2 \phi}{dx^2} + 5\phi = -\lambda \phi \quad \phi(0) = 0 \quad \phi(1) = 0. \]

5.8.12. Consider \( \partial^2 u/\partial t^2 = \alpha^2 \partial^2 u/\partial x^2 \) with the boundary conditions
\[ u(0) = u(L) = 0 \]
\[ \alpha \frac{\partial u}{\partial x} = -ku \quad \text{at } x = 0 \]
\[ m \frac{\partial^2 u}{\partial x^2} = -\alpha \frac{\partial u}{\partial x} - ku \quad \text{at } x = L. \]
(a) Give a brief physical interpretation of the boundary conditions.
(b) Show how to determine the frequencies of oscillation. Estimate the large frequencies of oscillation.
(c) Without attempting to use the Rayleigh quotient, explicitly determine if there are any separated solutions that do not oscillate in time. (Hint: There are none.)
(d) Show that the boundary condition is not self-adjoint: that is, show
\[ \int_0^L \left( u_{n} \frac{d^2 u_{n}}{dx^2} - u_{m} \frac{d^2 u_{m}}{dx^2} \right) dx \neq 0 \]
even when \( u_n \) and \( u_m \) are eigenfunctions corresponding to different eigenvalues.

*5.8.13. Simplify \( \int_0^L \sin^2 \sqrt{\lambda} x \, dx \) when \( \lambda \) is given by (5.8.15).

5.9 Large Eigenvalues (Asymptotic Behavior)
For the variable coefficient case, the eigenvalues for the Sturm-Liouville differential equation,
\[ \frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + \left[ \lambda \sigma(x) + q(x) \right] \phi = 0, \quad (5.9.1) \]
usually must be calculated numerically. We know that there will be an infinite number of eigenvalues with no largest one. Thus, there will be an infinite sequence of large eigenvalues. In this section we state and explain reasonably good approximations to these large eigenvalues and corresponding eigenfunctions. A careful derivation with adequate explanations of the asymptotic method would be lengthy. Nonetheless, some motivation for our result will be presented. We begin by attempting to approximate solutions of the differential equation (5.9.1) if the unknown eigenvalue \( \lambda \) is large (\( \lambda \gg 1 \)). Interpreting (5.9.1) as a spring-mass system (\( x \) is time, \( \phi \) is position) with time-varying parameters is helpful. Then (5.9.1) has a large restoring force \( -\lambda \sigma(x) \phi \) such that we expect the solution to have rapid oscillation in \( x \). Alternatively, we know that solutions corresponding to large eigenvalues have many zeros. Since the solution oscillates rapidly, over a few periods (a short distance) the variable coefficients are approximately constant. Thus, near any point \( x_0 \), the differential equation may be approximated crudely by one with constant coefficients:
\[ p(x_0) \frac{d^2 \phi}{dx^2} + \lambda \sigma(x_0) \phi = 0, \quad (5.9.2) \]
since in addition \( \lambda \sigma(x) \gg q(x) \). According to (5.9.2), the solution is expected to oscillate with "local" spatial (circular) frequency
\[ \text{frequency} = \sqrt{\frac{\lambda \sigma(x_0)}{p(x_0)}}, \quad (5.9.3) \]
This frequency is large (\( \lambda \gg 1 \)), and thus the period is small, as assumed. The frequency (and period) depends on \( x \), but it varies slowly; that is, over a few periods (a short distance) the period hardly changes. After many periods, the frequency (and period) may change appreciably. This slowly varying period will be illustrated in Fig. 5.9.1.

From (5.9.2) one might expect the amplitude of oscillation to be constant. However, (5.9.2) is only an approximation. Instead, we should expect the amplitude to be approximately constant over each period. Thus, both the amplitude and frequency are slowly varying:
\[ \phi(x) = A(x) \cos \phi(x), \quad (5.9.4) \]
where sines can also be used. The appropriate asymptotic formula for the phase \( \psi(x) \) can be obtained using the ideas we have just outlined. Since the period is small, only the values of \( x \) near any \( x_0 \) are needed to understand the oscillation implied by (5.9.4). Using the Taylor series of \( \phi(x) \), we obtain
\[ \phi(x) = A(x) \cos \left[ \phi(x_0) + (x - x_0) \phi'(x_0) + \cdots \right], \quad (5.9.5) \]
This is an oscillation with local frequency \( \psi'(x_0) \). Thus, the derivative of the phase is the local frequency. From (5.9.2) we have motivated that the local frequency should be \( (\lambda \sigma(x_0)/p(x_0))^{1/2} \). Thus, we expect
\[ \psi'(x_0) = \lambda^{1/2} \frac{\sigma(x_0)}{p(x_0)^{1/2}}, \quad (5.9.6) \]
This reasoning turns out to determine the phase correctly:
\[ \psi(x) = \lambda^{1/2} \int_{x_0}^x \frac{\sigma(x)}{p(x)^{1/2}} \, dx, \quad (5.9.7) \]
Note that the phase does not equal the frequency times \( x \) (unless the frequency is constant).
Precise asymptotic techniques beyond the scope of this text determine the slowly varying amplitude. It is known that two independent solutions of the differential equation can be approximated accurately (if \( \lambda \) is large) by

\[
\phi(x) \approx (\sigma p)^{-1/4} \exp \left[ \pm i \lambda^{1/2} \int \left( \frac{\sigma}{p} \right)^{1/2} \, dx_0 \right],
\]

(5.9.8)

where sines and cosines may be used instead. A rough sketch of these solutions (using sines or cosines) is given in Fig. 5.9.1. The solution oscillates rapidly. The envelope of the wave is the slowly varying function \((\sigma p)^{-1/4}\), indicating the relatively slow amplitude variation. The local frequency is \((\sigma x/p)^{-1/2}\), corresponding to the period \(2\pi/(\sigma x/p)^{1/2}\).

To determine the large eigenvalues, we must apply the boundary conditions to the general solution (5.9.8). For example, if \( \phi(0) = 0 \), then

\[
\phi(x) = (\sigma p)^{-1/4} \sin \left( \frac{\lambda^{1/2}}{p} \int_0^x \left( \frac{\sigma}{p} \right)^{1/2} \, dx_0 \right) + \cdots.
\]

(5.9.9)

The second boundary condition, for example, \( \phi(L) = 0 \), determines the eigenvalues

\[
0 = \sin \left( \frac{\lambda^{1/2}}{p} \int_0^L \left( \frac{\sigma}{p} \right)^{1/2} \, dx_0 \right) + \cdots.
\]

Thus, we derive the asymptotic formula for the large eigenvalues \(\lambda^{1/2} \int_0^L \left( \frac{\sigma}{p} \right)^{1/2} \, dx_0 \approx n\pi\), or, equivalently,

\[
\lambda \sim \frac{n\pi}{\int_0^L \left( \frac{\sigma}{p} \right)^{1/2} \, dx_0} \approx \frac{n^2\pi^2}{\int_0^L \left( \frac{\sigma}{p} \right)^{1/2} \, dx_0} \approx \frac{n^2}{3} \left( \frac{\sigma}{p} \right)^{1/2},
\]

(5.9.10)

valid if \( n \) is large. Often, this formula is reasonably accurate even when \( n \) is not very large. The eigenfunctions are given approximately by (5.9.9), where (5.9.10) should be used. Note that \( q(x) \) does not appear in these asymptotic formulas; \( q(x) \) does not affect the eigenvalue to leading order. However, more accurate formulas exist that take \( q(x) \) into account.

**Example.** Consider the eigenvalue problem

\[
\frac{d^2 \phi}{dx^2} + \lambda (1 + x) \phi = 0
\]

(5.9.11)

subject to the boundary conditions

(a) \( \phi(0) = 0 \) and \( \frac{d\phi}{dx}(L) = 0 \)

(b) \( \phi(0) = 0 \) and \( \frac{d\phi}{dx}(L) = 0 \)

(c) \( \phi(0) = 0 \) and \( \frac{d\phi}{dx}(L) + h\phi(L) = 0 \)

5.9.2. Consider

\[
\frac{d^2 \phi}{dx^2} + \lambda (1 + x) \phi = 0
\]

subject to \( \phi(0) = 0 \) and \( \phi(1) = 0 \). Roughly sketch the eigenfunctions for \( \lambda \) large. Take into account amplitude and period variations.
5.9.3. Consider for \( \lambda \gg 1 \)

\[
\frac{d^2 \phi}{dx^2} + [\lambda \sigma(x) + q(x)] \phi = 0.
\]

*(a) Substitute

\[
\phi = A(x) \exp \left[ i \lambda^{1/2} \int_0^x \sigma^{1/2}(x_0) \, dx_0 \right].
\]

Determine a differential equation for \( A(x) \).

(b) Let \( A(x) = A_0(x) + \lambda^{-1/2} A_1(x) + \cdots \). Solve for \( A_0(x) \) and \( A_1(x) \). Verify (5.9.8).

(c) Suppose that \( \phi(0) = 0 \). Use \( A_1(x) \) to improve (5.9.9).

(d) Use part (c) to improve (5.9.10) if \( \phi(L) = 0 \).

*(e) Obtain a recursion formula for \( A_n(x) \).

5.10 Approximation Properties

In many practical problems of solving partial differential equations by separation of variables, it is impossible to compute and work with an infinite number of terms of the series. It is more usual to use a finite number of terms. In this section, we briefly discuss the use of a finite number of terms of generalized Fourier series.

We have claimed that any piecewise smooth function \( f(x) \) can be represented by a generalized Fourier series of the eigenfunctions,

\[
f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x). \tag{5.10.1}
\]

Due to the orthogonality [with weight \( \sigma(x) \)] of the eigenfunctions, the generalized Fourier coefficients can be determined easily:

\[
a_n = \int_a^b f(x) \phi_n(x) \sigma(x) \, dx.
\]

However, suppose that we can only use the first \( M \) eigenfunctions to approximate a function \( f(x) \).

\[
f(x) \approx \sum_{n=1}^{M} a_n \phi_n(x). \tag{5.10.3}
\]

What should the coefficients \( a_n \) be? Perhaps if we use a finite number of terms, there would be a better way to approximate \( f(x) \) than by using the generalized Fourier coefficients, (5.10.2). We will pick these new coefficients \( a_n \) so that \( \sum_{n=1}^{M} a_n \phi_n(x) \) is the "best" approximation to \( f(x) \). There are many ways to define the "best," but we will show a way that is particularly useful. In general, the coefficients \( a_n \)

\section{Mean-square deviation.}

We define best approximation as the approximation with the least error. However, error can be defined in many different ways. The difference between \( f(x) \) and its approximation \( \sum_{n=1}^{M} a_n \phi_n(x) \) depends on \( x \). It is possible for \( f(x) - \sum_{n=1}^{M} a_n \phi_n(x) \) to be positive in some regions and negative in others. One possible measure of the error is the maximum of the deviation over the entire interval: namely \( \max_{x \in (a, b)} |f(x) - \sum_{n=1}^{M} a_n \phi_n(x)| \). This is a reasonable definition of the error, but it is rarely used, since it is very difficult to choose the \( a_n \) to minimize this maximum deviation. Instead, we usually define the error to be the mean-square deviation.

\[
E = \int_a^b \left[ f(x) - \sum_{n=1}^{M} a_n \phi_n(x) \right]^2 \sigma(x) \, dx. \tag{5.10.4}
\]

Here a large penalty is paid for the deviation being large on even a small interval. We introduce a weight factor \( \sigma(x) \) in our definition of the error because we will show that it is easy to minimize this error only with the weight \( \sigma(x) \). \( \sigma(x) \) is the same function that appears in the differential equation defining the eigenfunctions \( \phi_n(x) \); the weight appearing in the error is the same weight as needed for the orthogonality of the eigenfunctions.

The error, defined by (5.10.4), is a function of the coefficients \( a_1, a_2, \ldots, a_M \). To minimize a function of \( M \) variables, we usually use the first-derivative condition. We insist that the first partial derivative with respect to each \( a_i \) is zero:

\[
\frac{\partial E}{\partial a_i} = 0, \quad i = 1, 2, \ldots, M.
\]

We calculate each partial derivative and set it equal to zero:

\[
0 = \frac{\partial E}{\partial a_i} = -2 \int_a^b \left[ f(x) - \sum_{n=1}^{M} a_n \phi_n(x) \right] \phi_i(x) \sigma(x) \, dx, \quad i = 1, 2, \ldots, M. \tag{5.10.5}
\]

where we have used the fact that \( \partial/\partial a_i (\sum_{n=1}^{M} a_n \phi_n(x)) = \phi_i(x) \). There are \( M \) equations, (5.10.5), for the \( M \) unknowns. This would be rather difficult to solve, except for the fact that the eigenfunctions are orthogonal with the same weight \( \sigma(x) \) that appears in (5.10.5). Thus, (5.10.5) becomes

\[
\int_a^b f(x) \phi_i(x) \sigma(x) \, dx = a_i \int_a^b \phi_i^2(x) \sigma(x) \, dx.
\]
The $i$th equation can be solved easily for $a_i$. In fact, $a_i = a_i$, [see (5.10.2)]; all first partial derivatives are zero if the coefficients are chosen to be the generalized Fourier coefficients. We should still show that this actually minimizes the error (not just a local critical point, where all first partial derivatives vanish). We in fact will show that the best approximation (in the mean-square sense using the first $M$ eigenfunctions) occurs when the coefficients are chosen to be the generalized Fourier coefficients: In this way (1) the coefficients are easy to determine, and (2) the coefficients are independent of $M$.

**Proof.** To prove that the error $E$ is actually minimized, we will not use partial derivatives. Instead, our derivation proceeds by expanding the square deviation in (5.10.4):

$$E = \int_a^b \left( f^2 - 2 \sum_{n=1}^M a_n \phi_n + \sum_{n=1}^M a_n^2 \phi_n^2 \right) \sigma \, dx.$$  

(5.10.6)

Some simplification again occurs due to the orthogonality of the eigenfunctions:

$$E = \int_a^b \left( f^2 - 2 \sum_{n=1}^M a_n \phi_n + \sum_{n=1}^M a_n^2 \phi_n^2 \right) \sigma \, dx.$$  

(5.10.7)

Each $a_n$ appears quadratically:

$$E = \sum_{n=1}^M \left[ a_n^2 \int_a^b \phi_n^2 \sigma \, dx - 2a_n \int_a^b f \phi_n \sigma \, dx \right] + \int_a^b f^2 \sigma \, dx.$$  

(5.10.8)

and this suggests completing the square

$$E = \sum_{n=1}^M \left[ \int_a^b \phi_n^2 \sigma \, dx \left( a_n - \frac{\int_a^b f \phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx} \right)^2 - \left( \frac{\int_a^b f \phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx} \right)^2 \right] + \int_a^b f^2 \sigma \, dx.$$  

(5.10.9)

The only term that depends on the unknowns $a_n$ appears in a nonnegative way. The minimum occurs only if that first term vanishes, determining the best coefficients

$$a_n = \frac{\int_a^b f \phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx}.$$  

(5.10.10)

the same result as obtained using the simpler first derivative condition.

**Error.** In this way (5.10.9) shows that the minimal error is

$$E = \int_a^b f^2 \sigma \, dx - \sum_{n=1}^M \int_a^b \phi_n^2 \sigma \, dx.$$  

(5.10.11)

where (5.10.10) has been used. Equation (5.10.11) shows that as $M$ increases, the error decreases. Thus, we can think of a generalized Fourier series as an approximation scheme. The more terms in the truncated series that are used, the better the approximation.

**Example.** For a Fourier sine series, where $\sigma(x) = 1$, $\phi_n(x) = \sin n \pi x / L$ and $\int_0^L \sin^2 n \pi x / L \, dx = L / 2$, it follows that

$$E = \int_0^L f^2 \sigma \, dx = \frac{L}{2} \sum_{n=1}^M a_n^2.$$  

(5.10.12)

**Bessel's inequality and Parseval's equality.** Since $E \geq 0$ [see (5.10.4)], it follows from (5.10.11) that

$$\sum_{n=1}^M a_n^2 \int_a^b \phi_n^2 \sigma \, dx,$$  

(5.10.13)

known as Bessel's inequality. More importantly, we claim that for any Sturm-Liouville eigenvalue problem, the eigenfunction expansion of $f(x)$ converges in the mean to $f(x)$, by which we mean [see (5.10.4)] that

$$\lim_{M \to \infty} E = 0;$$

the mean-square deviation vanishes as $M \to \infty$. This shows Parseval's equality:

$$\int_a^b f^2 \sigma \, dx = \sum_{n=1}^\infty a_n^2 \int_a^b \phi_n^2 \sigma \, dx.$$  

(5.10.14)

Parseval's equality, (5.10.14), is a generalization of the Pythagorean theorem. For a right triangle, $c^2 = a^2 + b^2$. This has an interpretation for vectors. If $\mathbf{v} = a \mathbf{i} + b \mathbf{j}$, then $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = a^2 + b^2$. Here $a$ and $b$ are components of $\mathbf{v}$ in an orthogonal basis of unit vectors. Here we represent the function $f(x)$ in terms of our orthogonal eigenfunctions

$$f(x) = \sum_{n=1}^\infty a_n \phi_n(x).$$

If we introduce eigenfunctions with unit length, then

$$f(x) = \sum_{n=1}^\infty a_n \phi_n(x).$$
where \( l \) is the length of \( \phi_n(x) \):

\[
I^2 = \int_a^b \phi_n^2(x) \, dx.
\]

Parseval's equality simply states that the length of \( f \) squared, \( \int_a^b f^2 \, dx \), equals the sum of squares of the components of \( f \) (using an orthogonal basis of functions of unit length), \( (a_n)^2 = \int_a^b \phi_n^2(x) \, dx \).

**EXERCISES 5.10**

5.10.1. Consider the Fourier sine series for \( f(x) = 1 \) on the interval \( 0 \leq x \leq L \). How many terms in the series should be kept so that the mean-square error is 1% of \( \int_0^L f^2 \, dx \)?

5.10.2. Obtain a formula for an infinite series using Parseval's equality applied to the

(a) Fourier sine series of \( f(x) = 1 \) on the interval \( 0 \leq x \leq L \)

(b) Fourier cosine series of \( f(x) = x \) on the interval \( 0 \leq x \leq L \)

(c) Fourier sine series of \( f(x) = x \) on the interval \( 0 \leq x \leq L \)

5.10.3. Consider any function \( f(x) \) defined for \( a \leq x \leq b \). Approximate this function by a constant. Show that the best such constant (in the mean-square sense, i.e., minimizing the mean-square deviation) is the constant equal to the average of \( f(x) \) over the interval \( a \leq x \leq b \).

5.10.4. (a) Using Parseval's equality, express the error in terms of the tail of a series.
   
   (b) Redo part (a) for a Fourier sine series on the interval \( 0 \leq x \leq L \).

(c) If \( f(x) \) is piecewise smooth, estimate the tail in part (b).

   *(Hint: Use integration by parts.)*

5.10.5. Show that if

\[
L(f) = \frac{d}{dx} \left( p \frac{df}{dx} \right) + qf,
\]

then for normalized eigenfunctions

\[
\int_a^b f g \sigma \, dx = \sum_{n=1}^\infty \alpha_n \beta_n,
\]

a generalization of Parseval's equality.

5.10.7. Using Exercises 5.10.5 and 5.10.6, prove that

\[
-\sum_{n=1}^\infty \lambda_n \alpha_n^2 = -pf \left| \frac{df}{dx} \right|_a^b + \int_a^b \left[ p \left( \frac{df}{dx} \right)^2 - qf^2 \right] \, dx.
\]

*(Hint: Let \( g = L(f) \), assuming that term-by-term differentiation is justified.)*

5.10.8. According to Schwarz's inequality (proved in Exercise 2.3.10), the absolute value of the pointwise error satisfies

\[
\left| f(x) - \sum_{n=1}^M \alpha_n \phi_n \right| \leq \left\{ \sum_{n=M+1}^\infty \alpha_n^2 \right\}^{1/2} \left\{ \sum_{n=M+1}^\infty \lambda_n \right\}^{1/2}
\]

Furthermore, Chapter 9 introduces a Green's function \( G(x, x_0) \), which is shown to satisfy

\[
\sum_{n=1}^\infty \lambda_n \phi_n^2 = -G(x, x_0).
\]

Using (5.10.15), (5.10.16), and (5.10.17), derive an upper bound for the pointwise error (in cases in which the generalized Fourier series is pointwise convergent). Examples and further discussion of this are given by Weinberger [1995].
Chapter 6. Finite Difference Numerical Methods

6.7.5. Show (by completing square of quadratics) that the minimum of

\[ \int_R \left( \frac{1}{2} \nabla U)^2 - f(x,y)U \right) dA, \]

where \( U \) satisfies (6.7.3), occurs when \( KU = F \).

6.7.6. Consider a somewhat arbitrary triangle (as illustrated in Figure 6.7.5) with

\( P_1 = (0,0), P_2 = (L,0), P_3 = (D,H) \) and interior angles \( \theta_i \). The solution on the triangle will be linear \( U = a + bx + cy \).

Figure 6.7.5 Triangular finite element.

(a) Show that \( \int_R \left( \nabla U)^2 \right) dA = (b^2 + c^2) \frac{1}{2} LH \).

(b) The coefficients \( a, b, c \) are determined by the conditions at the three

vertices \( \sum_i U(P_i) = U_i \). Demonstrate that \( a = \frac{U_1, b = \frac{U_2-U_3}{L}, \) and \( c = \frac{U_3-U_1}{H} \).

(c) Show that \( \tan \theta_1 = \frac{B}{C}, \frac{1}{\tan \theta_2} = \frac{k_2}{k_3}, \frac{1}{\tan \theta_3} = \frac{k_3}{k_1} \), and using \( \tan \theta_1 = -\tan(\theta_1 + \theta_2) \), show that \( \frac{1}{\tan \theta_2} = \frac{b}{a + \theta_2} + \frac{c}{2} \).

(d) Using Exercise 6.7.4 and parts (a), (b), (c) of this exercise, show that for the contribution from this one triangle, \( K_{12} = \frac{1}{2} \tan \theta_5 \). The other entries of the stiffness matrix follow in this way.

6.7.7. Continue with part (d) of Exercise 6.7.6 to obtain

(a) \( K_{11} \)  \hspace{1cm} (b) \( K_{22} \)  \hspace{1cm} (c) \( K_{33} \)  \hspace{1cm} (d) \( K_{23} \)  \hspace{1cm} (e) \( K_{13} \)

Chapter 7

Higher Dimensional Partial Differential Equations

7.1 Introduction

In our discussion of partial differential equations, we have solved many problems by the method of separation of variables, but all involved only two independent variables:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

In this chapter we show how to extend the method of separation of variables to problems with more than two independent variables.

In particular, we discuss techniques to analyze the heat equation (with constant thermal properties) in two and three dimensions.

\[ \frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (two \ dimensions) \]

\[ \frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (three \ dimensions) \]

for various physical regions with various boundary conditions. Also of interest will be the steady-state heat equation, Laplace's equation, in three dimensions,

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \]