INTRODUCTION

Purpose. These notes intend to introduce four main notions from analysis and use these to provide practice in deciding the validity of mathematical arguments.

The notions mentioned above are convergence, continuity, connectedness, and completeness. It is assumed that the reader has had some experience with convergence of sequences of numbers and with the continuity of functions.

General comment: The greatest attention should be directed to understanding definitions. In particular, it is important to study and construct examples of objects that do and do not have the defined properties, until these properties become clear.

Statements and Proofs.

We will take the properties of the real numbers as given. So, for example, we will suppose that the product of two negative numbers is positive. As another example, we accept as known that between any two rational numbers there is an irrational number and that between any two irrational numbers there is a rational number.

Most of these notes consist of statements that the readers are asked to prove or disprove. We now follow with some examples of proofs and counter examples. These examples deal with the topic of the number of elements in a set. (I’ve chosen this topic because it is a good platform for illustrating methods of proof and because while the statements are simple, the proofs are not obvious – making it clear that some creativity and attention to detail is needed in constructing proofs).

Notation: The natural numbers are denoted here by $\mathbb{N}$ and are the set consisting of $1, 2, 3, ...$. The real numbers are denoted by $\mathbb{R}$, and the rational numbers by $\mathbb{Q}$. A mapping, $f$, from a set $A$ to a set $B$ is denoted by $f : A \rightarrow B$. We denote the difference of two sets by $A \setminus B$, that is $A \setminus B = \{a \in A : a \notin B\}$.

Definition 1. A mapping $f : A \rightarrow B$ is one-to-one or injective when $f(a) = f(x)$ implies $a = x$. A mapping $f : A \rightarrow B$ is onto or surjective when for any $b \in B$ there is an $a \in A$ with $f(a) = b$. A mapping $f : A \rightarrow B$ has a left inverse when there is a mapping $g : B \rightarrow A$ so that for any $a \in A$ it holds that $g \circ f(a) = a$. A mapping $f : A \rightarrow B$ has
a **right inverse** when there is a mapping \( g : B \to A \) so that for any \( b \in B \) it holds that \( f \circ g(b) = b \).

**Example 1.** A mapping \( f : A \to B \) which is one-to-one and onto has a left inverse.
Proof: Fix \( b \in B \). Since \( f \) is onto, there is an \( a \in A \) so that \( f(a) = b \). Moreover, since \( f \) is one-to-one, there is only one such \( a \). Set \( g(b) = a \). By the definition of \( g \), \( g \circ f(a) = a \).

This is an example of a **direct proof.** The proof started with the hypotheses and concluded with the desired consequence. The proof was actually constructive, having shown that \( g \) exists by giving its values (assuming the values of \( f \) are known).

**Example 2.** A mapping \( f : A \to B \) which is one-to-one and onto has a right inverse.
Proof: Define \( g \) as in the previous example. Fix \( b \in B \). Since \( f \) is onto, there is an \( a \in A \) so that \( f(a) = b \). Now \( f \circ g(b) = f \circ g(f(a)) \) and since \( g(f(a)) = a \), \( f \circ g(f(a)) = f(a) = b \). So \( f \circ g(b) = b \).

**Example 3.** Suppose a mapping \( f : A \to B \) has a left inverse \( g \). Then \( f \) is one-to-one.
Proof: Suppose that for two \( a_1, a_2 \in A \) we have \( f(a_1) = f(a_2) \). Then \( a_1 = g(f(a_1)) = g(f(a_2)) = a_2 \) and \( a_1 = a_2 \).

**Example 4.** Suppose a mapping \( f : A \to B \) has a right inverse \( h \). Then \( f \) is onto.
Proof: Fix \( b \in B \). Set \( a = h(b) \). Now \( f(a) = f(h(b)) = b \).

**Example 5.** Suppose a mapping \( f : A \to B \) has a left inverse \( g \) and a right inverse \( h \). Then \( g = h \).
Proof: The examples above show that \( f \) is one-to-one and onto. Fix \( b \in B \). Now \( f(h(b)) = b \) since \( h \) is a right inverse. Since \( f \) is onto, there is an \( a \in A \) with \( f(a) = b \), and \( f(g(b)) = f(g(f(a))) = f(a) = b \) (since \( g(f(a)) = a \)). Since \( f \) is one-to-one, \( g(b) = h(b) \).

Notation: We have seen that when a mapping \( f : A \to B \) is one-to-one and onto then it has a left inverse \( g \) and that \( g \) is also a right inverse for \( f \). This common function \( g \) is called the **inverse** of \( f \) and is denoted by \( f^{-1} \). Cultural note: the underlying operation for the power notation is composition of functions (rather than multiplication of functions).

**Example 6.** Decide whether the following statement is true: Suppose a mapping \( f : A \to B \) has a left inverse \( g \). Then \( f \) is one-to-one and onto.
Decision: This statement is false. To see this consider \( f : \mathbb{N} \to \mathbb{N} \) defined by \( f(n) = 2n \). This mapping is clearly not onto because it misses all the odd natural numbers. Set \( g(2m) = m \) and \( g(2k - 1) = k \). The mapping \( g : \mathbb{N} \to \mathbb{N} \) is defined everywhere and \( g \circ f(n) = n \). We have given a **counter-example** that shows the statement to be false.

**Definition 2.** An empty set is finite with zero elements. A non-empty set \( A \) is called **finite** and is said to have \( n \) elements when there is a mapping \( f : A \to \{1, 2, 3, \ldots, n\} \) which is one-to-one and onto. When for a fixed non-empty set \( A \) there is no such mapping (for any \( n \)), the set is called infinite. A set \( A \) is called **countable** when there is a mapping \( f : A \to \mathbb{N} \) which is one-to-one.

Notice that finite sets are automatically countable (including the empty set).
Example 7. A subset of a finite set is finite.
Proof: Let $A$ denote the finite set and let $B \subset A$. Then there is an $n$ and a one to one
and onto map $f : A \rightarrow \{1, \ldots, n\}$. Consider the image $f(B)$. If $B$ is the empty set, $\emptyset$, then $B$ is finite by definition. Suppose $B \neq \emptyset$. There is a least element, $i$, in $f(B)$ and a
$b_1$ so that $f(b_1) = i$. Set $g(b_1) = 1$. Continue in this fashion, that is, $b_1, \ldots, b_k$ are such
that $g(b_j) = j$ and $f(b_{j+1})$ is the smallest number in the image of $B \setminus \{b_1, \ldots, b_j\}$ under
$f$. The map $g$ is one to one and $g(b_j) \leq n$. Let $m$ be the maximum value in $g(B)$. Then
g : B \rightarrow \{1, \ldots, m\}$ is onto. To see this note that, by its definition, there are $b_1$ and $b_m$
with $g(b_1) = 1$ and $g(b_m) = m$, and if $i$ with $1 < i < m$ is not in the image of $g$ then
neither is $i - 1$.

Example 8. The set of natural numbers, $\mathbb{N}$ is infinite.
Proof: By contradiction (our first example of a proof using this method). Suppose there is
an $n$ and a one-to-one and onto map $f : \mathbb{N} \rightarrow \{1, 2, \ldots, n\}$. Set $a_i = f^{-1}(i)$. Notice that
for each $i$, $1 \leq i \leq n$, $a_i$ is well defined and a natural number. Let $k = a_1 + a_2 + \ldots + a_n$.
Then $k \in \mathbb{N}$ and $k \neq a_i$ for any $i$. However, $f(k) = j$ for some $j$ in $\{1, 2, \ldots, n\}$, so
$f(k) = f(a_j)$. This contradicts $f$ being one-to-one.

Example 9. The set of positive rational numbers is countable.
Proof: We will define a map from the positive rationals to $\mathbb{N}$ as follows. For each positive
rational number, $x$, there is a pair of positive natural numbers $(m, n)$ so that $x = m/n$
and $m$ and $n$ have no common factors. Set
$$f(x) = \frac{(m+n-1)(m+n-2)}{2} + m.$$  
This function is well defined, and we claim that it is one-to-one. To see this, notice that
if $k = m + n$ and $(i, j)$ is a pair of natural numbers with $i + j < k$, then $f(i, j) \leq
(k-2)(k-3)/2 + (k-2) = (k-1)(k-2)/2 < f(m, n)$ for such pairs. For those pairs
$(m, n)$ with $m + n = k$, say $(m_1, n_1) \neq (m_2, n_2)$, the values of $f$ differ. For instance,
f$(m_1, n_1) - f(m_2, n_2) = m_1 - m_2$. Hence $\mathbb{Q}$ is countable.
Note that we’ve shown that the set of pairs of natural numbers is countable (and have
shown that the positive rationals can be mapped into this set). Hence the proof actually
shows that the set product of two countable sets is countable. The formula used in the proof
to get a one-to-one map from $\mathbb{Q}$ to $\mathbb{N}$ is also not the main point. Rather, showing there is
a one-to-one map from a set into $\mathbb{N}$ is sufficient to conclude that the set is countable.

Example 10. The interval, $[0, 1]$ is not countable.
Proof: By contradiction. Suppose that there is a one-to-one map $f : [0, 1] \rightarrow \mathbb{N}$. For each
$i \in \mathbb{N}$ that is in the image of $f$ there is an $x_i$ in $[0, 1]$ with $f(x_i) = i$. Consider the decimal
expansion of $x_i$ that does not terminate in repeating 9s. This expansion is well-defined.
Let $d_i$ be the $i^{th}$ digit of $x_i$. Now define a number $z$ by specifying its decimal expansion
to have a digit different from $d_i$ for its $i^{th}$ digit. For example, the $i^{th}$ digit of $z$ could be
$|d_i - 1|$ if $i$ is in the image of $f$ and 3 otherwise. Notice that $z \in [0, 1]$ and so $f(z) = j$
for some \( j \). Notice that \( z \neq x_j \) for any \( j \). Also, if \( i \) is not in the image of \( f \), then clearly \( f(z) \neq i \). So \( f(z) = j \) for some \( j \) with \( x_j \) as above, which contradicts \( f \) being one-to-one.

Summary of introduction: We assume the usual properties of numbers. We have seen examples of direct proofs (including constructive proofs) and proofs by contradiction with both of these using logical arguments and definitions. We also saw how a counter-example can prove a statement false.

OPEN and CLOSED SETS

This section concerns sets in the real line, in the Euclidean plane, and in Euclidean 3-space.

**Definition** 3. A set \( O \) in the real line is **open** if for every point \( p \in O \) there is an \( r > 0 \) so that the interval \((p - r, p + r)\) is in \( O \). Symbolically, \( \exists r > 0 \) with \((p - r, p + r) \subset O \).

**Definition** 4. A set \( O \) in the plane or in \( \mathbb{R}^3 \) is **open** if for every point \( p \in O \) there is an \( r > 0 \) so that the disk (or ball) \( \{ x : |x - p| < r \} \) is in \( O \). Here \( |x - p| \) denotes the (Euclidean) distance between \( x \) and \( p \).

**Exercise** 1. Sketch each of the following sets in the real line (a rough sketch will do) and decide whether it is open.

(a) \( \{ x : |x - 1| < 2 \} \). (b) \( \{ x : |x - 1| \leq 2 \} \). (c) \( \{ x : |x - 1| = 2^{-n} \text{ for some } n \in \mathbb{N} \} \).

**Exercise** 2. Sketch each of the following sets in the real line (a rough sketch will do) and decide whether it is open.

(a) \( \{ x : |x - 1| < 2 \text{ or } |x - 3| < 1 \} \). (b) \( \{ x : |x - 1| \leq 2 \text{ and } |x - 3| < 1 \} \).

(c) \( \{ x : |x - 1| < 1 - 2^{-n} \text{ for some } n \in \mathbb{N} \} \). (d) \( \{ x : |x - 2^n| < 1 \text{ for some } n \in \mathbb{N} \} \).

**Prove or contradict** 1. The intersection of two open sets in the line or the plane is open. Stated with symbols, let \( A \) and \( B \) be two given open sets. Then \( A \cap B \) is open.

**Prove or contradict** 2. The union of two open sets in the line or the plane is open. Symbolically, let \( A \) and \( B \) be two given open sets. Then \( A \cup B \) is open.

**Prove or contradict** 3. If \( O \) is an open set in the line or in the plane then there is some positive number \( r \) (particular to \( O \)) so that for each and every \( p \in O \), \( \{ x : |x - p| < r \} \) is in \( O \).

**Prove or contradict** 4. If \( O \) is an open set in the plane and \( B \) is a set in the plane that is not open, then \( O \cup B \) cannot be open.

**Prove or contradict** 5. If \( A \) and \( B \) are sets in the plane that are not open, then \( A \cap B \) cannot be open.

**Definition** 5. An **indexed collection of sets** is an index set \( I \) and for each \( \alpha \in I \) a set \( S_\alpha \).
Prove or contradict 6. The intersection of an indexed collection of open sets in the plane is open. That is, suppose that for each \(\alpha \in I\), \(A_\alpha\) is open. Then \(\bigcap_{\alpha \in I} A_\alpha\) is open.

Prove or contradict 7. The union of an indexed collection of open sets in the plane is open.

Prove or contradict 8. Let \(O\) be a set in the line or in the plane. If \(O\) is open then for each \(p \in O\) there is a rational number \(q > 0\) so that \(\{x : |x - p| < q\}\) is in \(O\). Conversely, if for each \(p \in O\) there is a rational number \(q > 0\) so that \(\{x : |x - p| < q\}\) is in \(O\), then \(O\) is open.

Definition 6. An open set containing a point \(p\) is called a neighborhood of \(p\).

Definition 7. In the line or the plane, a point \(p\) is an accumulation point of a set \(S\) when every neighborhood of \(p\) contains a point in \(S\) without considering \(p\). That is, every neighborhood of \(p\), even after \(p\) is deleted from this neighborhood, contains a point in \(S\).

Notation. For convenience we will denote the set of accumulation points of a set \(S\) by \(L(S)\). (This is a script \(L\), since these points are limit points, in some sense.)

Prove or contradict 9. Suppose that \(p\) is an accumulation point of \(A \cup B\). Then \(p\) is an accumulation point of \(A\) or \(p\) is an accumulation point of \(B\) (or both).

Definition 8. A set \(C\) in the line or the plane is closed when \(C\) contains all the accumulation points of \(C\).

Exercise 3. Sketch each of the following sets in the real line (a rough sketch will do) and decide whether it is closed.

(a) \(\{x : |x - 1| < 2\}\). (b) \(\{x : |x - 1| \leq 2\}\). (c) \(\{x : |x - 1| = 2^{-n}\text{ for some }n \in \mathbb{N}\}\).

Exercise 4. Sketch each of the following sets in the real line (a rough sketch will do) and decide whether it is closed.

(a) \(\{x : |x - 1| \leq 2\text{ or }|x - 3| \leq 1\}\). (b) \(\{x : |x - 1| < 2\text{ and }|x - 3| \leq 1\}\).

(c) \(\{x : |x - 1| \leq 1 - 2^{-n}\text{ for some }n \in \mathbb{N}\}\). (d) \(\{x : |x - 2^n| \leq 1\text{ for some }n \in \mathbb{N}\}\).

Prove or contradict 10. The intersection of two closed sets in the line or the plane is closed. Stated with symbols, let \(A\) and \(B\) be two given closed sets. Then \(A \cap B\) is closed.

Prove or contradict 11. The union of two closed sets in the line or the plane is closed. Symbolically, let \(A\) and \(B\) be two given closed sets. Then \(A \cup B\) is closed.

Prove or contradict 12. The intersection of an indexed collection of closed sets in the plane is closed.

Prove or contradict 13. The union of an indexed collection of closed sets in the plane is closed.

Prove or contradict 14. If \(C\) is a closed set in the plane and \(B\) is a set in the plane that is not closed, then \(C \cup B\) cannot be closed.

Prove or contradict 15. If \(A\) and \(B\) are sets in the plane that are not closed, then \(A \cap B\) cannot be closed.
Exercise 5. Decide whether the intersection of an open set and a closed set can be closed, open, or neither.

Exercise 6. Decide whether the union of an open set and a closed set can be closed, open, or neither.

Exercise 7. Decide whether there are sets that are both open and closed (at the same time).

Prove or contradict 16. Let $X$ denote the line or plane. If $O$ is an open set in $X$, then $X \setminus O$, the complement of $O$, is closed.

Prove or contradict 17. Let $X$ denote the line or plane. If $C$ is a closed set in $X$, then $X \setminus C$, the complement of $C$, is open.

Prove or contradict 18. Suppose $C$ is closed and $O$ is open. Then $C \setminus O$ is closed and $O \setminus C$ is open.

Definition 9. The boundary of a set $S$, denoted $\partial S$, consists of all points $p$ so that each neighborhood of $p$ contains both a point in $S$ and a point not in $S$.

Exercise 8. For each of the following sets in the plane sketch it (a rough sketch will do) and describe its boundary.

(a) $\{x : \|x - (1,2)\| < 2\}$. (b) $\{x : \|x - (3,1)\| \leq 2\}$. (c) $\{x : \|x - (3,3)\| = 2^{-n}$ for some $n \in \mathbb{N}\}$.

Prove or contradict 19. A set always contains its boundary.

Prove or contradict 20. A closed set always contains its boundary.

Prove or contradict 21. If a set contains its boundary, then it is closed.

Prove or contradict 22. The boundary of a union of two sets in the plane is contained in the union of the boundaries of the sets. That is, $\partial(A \cup B) \subset (\partial A) \cup (\partial B)$.

Prove or contradict 23. The boundary of an intersection of two sets in the plane is contained in the intersection of the boundaries of the sets. That is, $\partial(A \cap B) \subset (\partial A) \cap (\partial B)$.

Prove or contradict 24. A set with its boundary removed is open. (That is, if $A$ is a set in the line or the plane, then $A \setminus \partial A$ is open.)

Prove or contradict 25. The boundary of a set is a closed set. That is, $\partial A$ is always a closed set.

Definition 10. The closure of a set $S$, denoted $\overline{S}$, consists of all points $p$ so that either $p$ is in $S$ or $p$ is an accumulation point of $S$.

Prove or contradict 26. The closure of a set is a closed set.

Prove or contradict 27. The closure of a set consists of the set and the boundary of the set. That is, $\overline{S} = S \cup (\partial S)$.

Prove or contradict 28. If $C$ is closed and $S \subseteq C$, then $\overline{S} \subseteq C$. 


Prove or contradict 29. The closure of a set is the intersection of all closed sets that contain the original set. That is $\overline{S} = \cap \{C : C \text{ is closed and } S \subseteq C\}$.

**CONNECTED SETS**

This section concerns sets in the real line and in the plane. Similar definitions apply to sets in other settings.

**Definition** 11. Two sets are disjoint if their intersection is empty. That is $A$ and $B$ are disjoint when $A \cap B = \emptyset$.

**Definition** 12. Two non-empty sets $A$ and $B$ are separated if there are two open sets $O$ and $G$ so that $A \subseteq O$ and $B \subseteq G$ and $O \cap G = \emptyset$.

**Exercise** 9. For each of the following pairs of sets in the line decide whether they are separated.
(a) $A = \{x : |x - 1| < 2\}, B = \{x : |x - 4| \leq 1\}$.
(b) $A = \{x : |x - 1| < 2\}, B = \{x : |x - 4| < 1\}$.
(c) $A = \{x : x = 2^{-n} \text{ for some } n \in \mathbb{N}\}, B = \{x : x \leq 0\}$.

**Exercise** 10. For each of the following pairs of sets in the plane decide whether they are separated.
(a) $A = \{(x, y) : x^2 + y^2 < 1\}, B = \{(x, y) : (x - 2)^2 + y^2 \leq 1\}$.
(b) $A = \{(x, y) : x^2 + y^2 < 1\}, B_r = \{(x, y) : \sqrt{0.5} < x < 2, r < y < r + r^2\}$, where all $r > 0$ are to be considered.
(c) $A = \{(x, y) : x^2 + y^2 < 1\}, B = \{(x, y) : x^2 + y^2 > 1\}$.

**Exercise** 11. For each of the following pairs of sets in the line decide whether they are separated.
(a) $A$ consists of rational numbers in $[0, 1]$, $B$ consists of irrational numbers in $[0, 1]$.
(b) $A = \{x : x = n + 2^{-n} \text{ for some } n \in \mathbb{N}\}, B = \{x : x = n - 2^{-n} \text{ for some } n \in \mathbb{N}\}$.

**Exercise** 12. For each of the following pairs of sets in the plane decide whether they are separated.
(a) $A = \{(x, y) : x = 0, -1 \leq y \leq 1\}, B = \{(x, y) : 0 < x < 1, y = \sin(1/x)\}$.
(b) $A = \{(x, y) : x = 0, 0 \leq y \leq 1\}, B = \{(x, y) : x = 1/n, \text{ for some } n \in \mathbb{N}, 0 \leq y \leq 1\}$.
(c) $A = \{(x, y) : x^2 + y^2 = 1\}$,
$$B = \{(x, y) : x = (1 - e^{-t}) \cos(t), y = (1 - e^{-t}) \sin(t), \text{ for some } t \geq 0\}.$$

Prove or contradict 30. If $\overline{A} \cap B = \emptyset$ then $A$ and $B$ are separated.

Prove or contradict 31. Suppose $\overline{B} \cap A = \emptyset$. Set $D = \{x : x \notin \overline{B}\}$. Then for each $p \in (\partial A) \cap D$ there is an $r > 0$ so that $\{x : |x - p| < 2r\} \subseteq D$ and so that for $y \in B$, $|y - p| > 2r$.

Prove or contradict 32. Suppose $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. Set $D = \{x : x \notin \overline{B}\}$ as above, and for each $p \in (\partial A) \cap D$ let $r_p > 0$ be as in the previous statement. Set
\[ U = \cup \{ x : |x - p| < r, p \in (\partial A) \cap D \} \]. Then \( U \) is open. Let \( E = \{ x : x \notin \overline{A} \} \). Then one can construct an open set \( V \) similar to \( U \) but with the union over all points \( q \in (\partial B) \cap E \). Moreover \( U \cap V = \emptyset \).

**Prove or contradict** 33. Suppose \( A \cap B = \emptyset \) and \( \overline{A} \cap A = \emptyset \). Then \( A \) and \( B \) are separated.

**Prove or contradict** 34. If \( A \) and \( B \) are closed disjoint sets then they are separated.

**Definition** 13. A set \( S \) is **connected** if there are no two sets \( A \) and \( B \), both non-empty, so that \( S = A \cup B \) and \( A \) and \( B \) are separated. Such a pair \( \{A, B\} \) is called a separation of \( S \).

**Exercise** 13. For each of the following sets decide whether they are connected.
(a) A set consisting of exactly two points (in the line).
(b) \( A = \{ x : x = 2^{-n} \text{ for some } n \in \mathbb{N} \} \).
(c) An interval of the form \([a, b]\).

**Exercise** 14. For each of the following sets decide whether they are connected.
(a) The rational numbers in \([0, 1]\) of the form \( 1/n \) for some \( n \).
(b) The rational numbers in \([0, 1]\).

**Exercise** 15. Consider the sets \( S_r \) below and decide, for each \( r > 0 \), whether the set is connected.
\[ S_r = \{ (x, y) : x^2 + y^2 < 1 \} \cup \{ (x, y) : (x - 3)^2 + y^2 < 1 \} \cup \{ (x, y) : 0 < x < 3, 0 < y < r \} \].

**Prove or contradict** 35. Suppose \( A \) and \( B \) are connected sets. Assume that \( A \cap B \) is not empty. Then \( A \cap B \) is connected.

**Prove or contradict** 36. Suppose \( S \) is a connected set. Then \( \partial S \) is connected.

**Prove or contradict** 37. Suppose \( S \) is a connected set. Then \( \overline{S} \) is connected.

**Prove or contradict** 38. Suppose \( A \) and \( B \) are connected sets. Suppose \( A \cap B \neq \emptyset \). Then \( A \cup B \) is connected.

**Prove or contradict** 39. Suppose \( A_{\alpha} \) is a connected set for each \( \alpha \in I \). Suppose there is a point \( p \) so that \( p \in A_{\alpha} \) for each \( \alpha \in I \). Then the union \( \cup \{ A_{\alpha} : \alpha \in I \} \) is connected.

**Prove or contradict** 40. Suppose \( A_{\alpha} \) is a connected set for each \( n \in \mathbb{N} \). Suppose that for each \( n \) one has \( A_{n+1} \subset A_n \). Then the intersection \( \cap \{ A_n : n \in \mathbb{N} \} \) is connected.

**Least Upper Bound**

This section concerns sets in the real line. The least upper bound axiom is a property of real numbers.

**Definition** 14. A set \( S \subset \mathbb{R} \) has \( u \) as an **upper bound** if \( u \) is as large as any number in \( S \). Symbolically, \( s \leq u, \forall s \in S \).
**Definition** 15. A set $S \subset \mathbb{R}$ has **least upper bound** $b$ if $b$ is the smallest upper bound for $S$. That is, $b$ is an upper bound for $S$ and for any other upper bound, $u$, $b \leq u$.

**Exercise** 16. Find three different upper bounds for each of the following sets. Then find the least upper bound for each set.

(a) $S = \{x : x = 1/n, \text{ some positive integer } n\}$.
(b) $S = \{x : x < 2\}$.
(c) $S = \{x : x = 3 - 1/n, \text{ some positive integer } n\}$.

**Prove or contradict** 41. If a set has a least upper bound, then this number is unique. (Symbolically, if $b$ and $r$ are both least upper bounds for $S$ then $b = r$.)

We will assume that every set in the real line with an upper bound has a least upper bound. This is called the **least upper bound axiom**. In the usual formulations of the axioms for real numbers either this axiom, or an alternative one – that the real line is connected – is assumed.

**Definition** 16. A set $S \subset \mathbb{R}$ is an **interval** if it is convex. That is, for any $a, b \in S$ and any number $t$ with $0 \leq t \leq 1$, we have that $ta + (1-t)b \in S$.

**Definition** 17. A set $S \subset \mathbb{R}$ is **bounded** if there is a number $r$ with $S \subset [-r, r]$.

**Exercise** 17. Show that any bounded set has an upper bound.

**Exercise** 18. Define the **lower bound** of a set $S$ in the real line.

**Exercise** 19. Define the **greatest lower bound** of a set $S$ in the real line.

**Prove or contradict** 42. Any interval in the real line is connected.

**Prove or contradict** 43. Every connected set in the real line is an interval.

**Definition** 18. A set $S \subset \mathbb{R}$ has a **maximum** $x$ if $x \in S$ and $x$ is as large as any number in $S$. A set $S \subset \mathbb{R}$ has a **minimum** $x$ if $x \in S$ and $x$ is as small as any number in $S$.

**Exercise** 20. Show that not every bounded interval in the real line has a maximum.

**Exercise** 21. Show that if a set in the real line has a maximum, then it has a least upper bound.

Cultural note: The least upper bound of a set in the real line is also called the **supremum** of the set.

---

**SEQUENCES**

A sequence of real numbers or of points in the plane is an indexed collection $p_n$ of numbers or points with $n$ a natural number ($n$ is one of 1, 2, 3, ...).

**Definition** 19. A **sequence** of real numbers is a function from the natural numbers to the real numbers. A sequence of points in the plane is a function from the natural numbers
to the plane. The values of the function are called the terms of the sequence (and are ordered).

**Examples** 11. (a) \( x_n = n \), (b) \( p_n = (1, 1/n^2) \), (c) \( x_n = \sqrt{n} \), (d) \( p_n = (n, n^2) \), (e) \( x_n = 2^n \), (f) \( x_n = 0.4^n \), (g) \( p_n = (1/n, \sin(n)) \), (h) \( x_n = (-1)^n \), (i) \( p_n = (1/n, \sin(1/n)) \), (j) \( x_n = \cos(n)/n^2 \), (k) \( x_n = (n + 3)/(5n - 1) \).

Since the order of terms in a sequence is included, we often refer to the value of the term that appears in the \( n \)th position as the \( n \)th term.

Notation: We write \( \{x_n\} \) to denote the sequence whose \( n \)th term is \( x_n \). In other words, this refers to a sequence \( x : \mathbb{N} \to \mathbb{R} \) or \( x : \mathbb{N} \to \mathbb{R}^2 \) with \( x_n = x(n) \).

**Definition 20.** A sequence \( \{x_n\} \) **approaches a limit** \( v \) if given any positive number \( \epsilon > 0 \), there is an \( N \) so that whenever \( n \geq N \) we also have \( |x_n - v| < \epsilon \). When a sequence approaches some limit, it is said to **converge**.

**Exercise 22.** Which of the examples at the beginning of this section approaches a limit? and if so, what value or point does it approach?

**Prove or contradict** 44. If a sequence of real numbers converges (that is, it approaches a limit), then the set consisting of its terms is bounded.

**Prove or contradict** 45. If a sequence of real numbers does not converge, then the set consisting of its terms is not bounded.

**Prove or contradict** 46. If a sequence of real numbers converges, then the set consisting of its terms has an accumulation point.

**Prove or contradict** 47. If the set consisting of the terms of a sequence of real numbers has an accumulation point, then the sequence converges.

**Prove or contradict** 48. Suppose \( \{x_n\} \) is a sequence with \( x_n \in S \) for some set \( S \). Suppose that the sequence converges to \( v \), with \( v \notin S \). Then \( S \) is not closed.

**Definition 21.** A **subsequence** of a sequence \( \{x_n\} \) is a sequence obtained by skipping over some of the terms in \( \{x_n\} \). More carefully, let \( x_n = f(n) \). A subsequence is given by \( g : \mathbb{N} \to \mathbb{N} \) which is a strictly increasing function (that is \( g(i + 1) > g(i) \)) through the rule \( y_i = f(g(i)) \).

Notation: We often write \( x_{n_i} \) for the \( i \)th term of the subsequence. That is, for \( f : \mathbb{N} \to \mathbb{R} \), and a strictly increasing \( g : \mathbb{N} \to \mathbb{N} \), we think of the terms as \( x_n = f(n) \) and \( n_i = g(i) \), and of the sequence and subsequence as \( \{x_n\} \) and \( \{x_{n_i}\} \).

**Exercise 23.** Set \( x_n = f(n) = (-1)^n \). Let \( g(i) = 2i \) and \( h(j) = 2j - 1 \). Describe the terms of the subsequences \( f(g(i)) \) and \( f(h(j)) \).

**Exercise 24.** Set \( x_n = f(n) = [(-1)^n/n] + [(-1)^{n+1}/(n + 1)] \). Let \( g(i) = 2i \) and \( h(j) = 2j - 1 \). Describe the terms of the subsequences \( f(g(i)) \) and \( f(h(j)) \).

**Exercise 25.** If \( n \) is odd let \( x_n = 1/n \), and if \( n \) is even let \( x_n = 7 \). Find a subsequence of \( \{x_n\} \) that converges and a subsequence of \( \{x_n\} \) that does not converge. (You should be able to find more than one of each.)
Definition 22. A sequence of real numbers \( \{x_n\} \) is increasing if \( x_{n+1} \geq x_n \) for each \( n \) (so the function \( f \) in the definition of a sequence is an increasing function). A sequence \( \{x_n\} \) is strictly increasing if \( x_{n+1} > x_n \). A sequence \( \{x_n\} \) is decreasing if \( x_{n+1} \leq x_n \). A sequence \( \{x_n\} \) is strictly decreasing if \( x_{n+1} < x_n \).

Prove or contradict 49. If a sequence of real numbers has a strictly increasing subsequence, then the (original) sequence cannot be decreasing.

Prove or contradict 50. If a sequence of real numbers converges and has a strictly increasing subsequence, then the (original) sequence is increasing.

Prove or contradict 51. An increasing sequence of real numbers \( \{x_n\} \) with \( (x_{n+1} - x_n) \to 0 \) (as \( n \to \infty \)) converges.

Prove or contradict 52. Suppose that a sequence of real numbers is increasing and has a convergent subsequence. Then the sequence converges.

Prove or contradict 53. Suppose that a sequence of real numbers is increasing and has an upper bound (that is, the set of terms has an upper bound). Then the sequence converges to its least upper bound.

Intuitively, the first few terms of a sequence do not determine whether it converges. We will now turn our attention to the “tail end” of sequences. That is, for a fixed \( m \) we think of the terms \( \{x_n\} \) where \( n \geq m \). Of course, the beginning of the end (that is \( m \)) will vary.

Prove or contradict 54. A sequence of real numbers that does not have an upper bound has an increasing subsequence. Similarly, a sequence of real numbers that does not have a lower bound has a decreasing subsequence.

Prove or contradict 55. Fix a sequence of real numbers \( \{x_n\} \). Suppose that the sequence is bounded and let \( u \) be the least upper bound of the collection \( \{x_1, x_2, \ldots\} \). Suppose that \( u \neq x_n \) for any \( n \). Let \( y_j = \max\{x_1, x_2, \ldots, x_j\} \). Then for any \( k > j \), \( y_k \geq y_j \) and for any fixed \( j \) there is an \( m > j \) so that \( y_m > y_j \).

Prove or contradict 56. Fix a sequence of real numbers \( \{x_n\} \). Suppose this sequence is bounded. Let \( l_j \) be the least upper bound of \( \{x_j, x_{j+1}, x_{j+2}, \ldots\} \). Then (a) \( \{l_j\} \) is a decreasing sequence, and (b) it is possible that \( l_j \) is a subsequence of \( \{x_n\} \) but also that it is not a subsequence of \( \{x_n\} \).

Prove or contradict 57. Every sequence of real numbers has either an increasing subsequence or a decreasing subsequence.

Prove or contradict 58. Every bounded sequence of real numbers has a convergent subsequence. That is, suppose there are lower and upper bounds for the terms of a sequence, say \( a \leq x_n \leq b \), then \( \{x_n\} \) has a subsequence that converges.

CAUCHY SEQUENCES
Definition 23. A sequence \( \{ x_n \} \) is Cauchy (or a Cauchy sequence) if given any \( \epsilon > 0 \) there is a \( K \) so that if \( m, n \geq K \) then \( |x_n - x_m| < \epsilon \). Note that \( K \) may depend on \( \epsilon \).

Exercise 26. Decide which of the following sequences is a Cauchy sequence.

(a) \( x_n = \frac{1}{n} \),  
(b) \( x_n = \ln(n) \),  
(c) \( x_n = \frac{1 + (-1)^n}{5} \),  
(d) \( x_n = \frac{0.3^n}{3 + n + (-1)^n n} \).

Prove or contradict 59. Every Cauchy sequence of real numbers is bounded.

Prove or contradict 60. Every convergent sequence of real numbers is Cauchy.

Prove or contradict 61. Every Cauchy sequence of real numbers converges.

Definition 24. A set on which a distance is define is called a metric space and such a metric space is complete if every Cauchy sequence in the space converges.

Prove or contradict 62. Every Cauchy sequence whose terms are in \((0, 1)\) converges to a limit \( v \) with \( v \in (0, 1) \).

Comments: We did not define what we mean by a distance, but an example with which the reader is familiar is the Euclidean distance in two or three dimensions. The problems above suggest that the real numbers is a complete metric space. The idea is that a Cauchy sequence discovers whether there are holes in the space – that a sequence is Cauchy tells us that it should have a limit.

COMPACT SETS

We consider only sets in the real line in this section. However, the reader might consider how the objects defined here generalize to the plane and to other spaces.

Definition 25. An indexed collection of open sets, \( O_\alpha \) with \( \alpha \in I \), is an open cover of the set \( S \) when \( S \subset \bigcup_\alpha O_\alpha \).

Exercise 27. Fix a set \( S \) (in the real line) and a number \( \delta > 0 \). Consider the collection \( O_s = (s - \delta, s + \delta) \) with \( s \in S \). Is this an open cover of \( S \)?

Definition 26. An open cover of \( S \), \( O_\alpha \) with \( \alpha \in I \), has a finite sub-cover if there is a finite \( n \) so that \( S \subset O_{\alpha_1} \cup \ldots \cup O_{\alpha_n} \). That is, there are a finite number of elements \( \alpha_i \in I \) so that \( S \) is contained in the union of the corresponding sets.

Exercise 28. Let \( S = (0, 1] \) and for each integer \( i \geq 10 \) set \( O_i = (1/i, 1 + 1/i) \). Show that the collection of \( O_i \) is an open cover for \( S \). Is there a finite sub-cover of this cover?

Exercise 29. Let \( S = [0, 1] \). Fix \( \delta > 0 \) and set \( O_i = (-\delta + 1/i, \delta + 1/i) \) for \( i \in \mathbb{N} \). For which \( \delta \) is the collection of \( O_i \) an open cover for \( S \). Is there a finite sub-cover of this cover?

Exercise 30. Let \( S = [0, 1] \) and set

\[
O_{i,n} = \left( \frac{i}{2^n} - \frac{1}{n+3}, \frac{i}{2^n} + \frac{1}{n+3} \right), \quad i, n \in \mathbb{N}, \quad i < 2^n.
\]
Show that the collection of $O_{i,n}$ are an open cover for $S$. Is there a finite sub-cover of this cover?

**Definition 27.** A set $S$ is **compact** if every open cover of $S$ has a finite sub-cover.

**Exercise 31.** Is a set of finitely many real numbers compact? Is $\mathbb{N}$ compact?

**Exercise 32.** Is the open interval $(0,1)$ compact?

**Prove or contradict 63.** Every compact set is bounded.

**Prove or contradict 64.** A union of two compact sets is compact.

**Prove or contradict 65.** Any sequence in a compact set has a convergent subsequence.

**Prove or contradict 66.** A closed subset of a compact set is compact. That is, if $C \subset S$ and $C$ is closed and $S$ is compact, then $C$ is compact.

**Prove or contradict 67.** Any sequence in a compact set converges.

**Prove or contradict 68.** Any compact set is closed. Suggestion: let $v$ be an accumulation point of a compact set $S$. Construct a collection of open sets that would be a cover of $S$ with no finite sub-cover if $v \not\in S$.

**Prove or contradict 69.** The intersection of two compact sets is compact.

**Prove or contradict 70.** Suppose that a set $S$ has the property that every infinite sequence in $S$ has a subsequence that converges. Then $S$ is bounded.

**Prove or contradict 71.** Suppose that a set $S$ has the property that every infinite sequence in $S$ has a subsequence that converges to a point in $S$. Then $S$ is closed.

**Prove or contradict 72.** The set $[a,b]$ is compact.

**Prove or contradict 73.** If a set (in $\mathbb{R}$) is bounded and closed then it is compact.

**Prove or contradict 74.** A set $S$ is countably compact if every countable open cover of $S$ has a finite sub-cover. A countably compact set is compact.

**Example 12.** The Cantor set is defined as follows. The set is a subset of $[0,1]$.

- Set $C_1 = [0,1] \setminus (1/3, 2/3) = [0,1/3] \cup [2/3,1]$. Notice that $C_1$ is closed and bounded.
- Set $C_2 = C_1 \setminus ((1/9, 2/9) \cup (7/9, 8/9))$.
- Set $C_3 = C_2 \setminus ((1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20/27) \cup (25/27, 26/27))$.

Continue in this fashion, with $C_k$ being the union of $2^k$ disjoint closed intervals, and with the middle third of each of these intervals removed to form $C_{k+1}$.

The cantor set is $C = \cap C_i$, the intersection being over all $i \in \mathbb{N}$.

The cantor set is closed because it is the intersection of closed sets, and it is bounded because it is contained in $[0,1]$. Hence $C$ is compact.

When considering the size of the Cantor set, the following is interesting. The base 3 expansion of $x \in [0,1]$ is $x = \sum d_i 3^{-i}$ where the sum is over $i \in \mathbb{N}$ and the digits are $d_i \in \{0,1,2\}$. (For some numbers the representation is not unique.)

**Exercise 33.** Write the endpoints of $C_1$, $C_2$, and $C_3$ using the base 3 expansion.
Exercise 34. How does one obtain the base 3 expansion of the endpoints to $C_{n+1}$ from the base 3 expansion of the endpoints to $C_n$?

Example 13. The Cantor set consists of those points in $[0, 1]$ that have a base 3 expansion using only the digits 0 and 2.

CONTINUOUS FUNCTIONS

Here we consider a real valued function whose domain is a subset of the real numbers. We aim to understand the consequences of continuity for a function defined on a compact set in $\mathbb{R}$.

Definition 28. A function $f$ defined on a domain $D$ is continuous at a point $p$ in $D$ if given any $\epsilon > 0$ there is a $\delta > 0$ so that $f$ is defined on $(p - \delta, p + \delta)$ and $|f(x) - f(p)| < \epsilon$ whenever $|x - p| < \delta$.

Exercise 35. Show that $f(x) = x^3$ is continuous at any point $p \in \mathbb{R}$.

Exercise 36. Show that $f(x) = 1/x$ is continuous at $p = 2$ (in fact it is continuous at any point $p \neq 0$).

Prove or contradict 75. The function $f(x)$ defined below is continuous at any point $p$ that is irrational ($p \not\in \mathbb{Q}$), and is not continuous at any point $p$ that is rational.

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{m}{n} \text{ in lowest terms} \\ 0 & x \text{ irrational} \end{cases}$$

Hint: For any fixed $x$ and $n$ there are only finitely many rational numbers with denominator at most $n$ inside, say, $(x - 1, x + 1)$.

Prove or contradict 76. The sum of two continuous functions (defined on the same domain) is continuous.

Prove or contradict 77. Suppose $f : D \to \mathbb{R}$ is continuous at $p$. Suppose $\{x_n\}$ with $x_n \in D$ has $x_n \to p$. Then $f(x_n) \to f(p)$.

Prove or contradict 78. Suppose $f : D \to \mathbb{R}$ where $D$ is an open set. Fix $p \in D$. Suppose that for any sequence $\{x_n\}$ with $x_n \in D$ and $x_n \to p$ one has $f(x_n) \to f(p)$. Then $f$ is continuous at $p$.

Definition 29. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $S$ be a set (in $\mathbb{R}$). Then the image of $S$ under $f$ is $f(S) = \{y : y = f(x), \text{ for some } x \in S\}$. The inverse image of $S$ under $f$ is $f^{-1}(S) = \{x : y = f(x), \text{ for some } y \in S\}$.

Prove or contradict 79. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and let $O$ be an open set. Then the inverse image of $O$, $f^{-1}(O)$, is open.

Prove or contradict 80. Suppose $f : \mathbb{R} \to \mathbb{R}$ and that for any open set $O$ it is known that $f^{-1}(O)$ is open. Then $f$ is continuous.
**Terminology:** A function \( f \) has a **maximum value** \( M \) on a set \( S \) and the maximum value is **achieved at** a point \( p \) when \( f(p) \geq f(x) \) for any \( x \in S \).

**Exercise 37.** What is the maximum value of \( f(x) = x^3 \) on \([0, 1]\)?

**Exercise 38.** Is there a maximum value for \( f(x) = x^3 \) on \((0, 1)\)?

**Exercise 39.** What is the least upper bound for the values of \( f(x) = x^3 \) on \((0, 1)\)?

**Terminology:** We call the least upper bound for the values of \( f \) on the set \( S \) the **supremum** of \( f \) on \( S \).

**Definition 30.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function and let \( \{x_n\} \) be a sequence (in \( \mathbb{R} \)). Then \( f(x_n) \to \infty \), the function **approaches infinity** on this sequence, if given any \( B \) there is an \( M \) so that \( f(x_n) > B \) for all \( n \) with \( n \geq M \).

**Prove or contradict 81.** Assume that \( f \) is defined on a set \( S \) and suppose that the values of \( f \) on \( S \) have no upper bound. Then there is a sequence \( \{x_n\} \) with \( x_n \in S \) so that \( f(x_n) \to \infty \).

**Prove or contradict 82.** Assume that \( f \) is defined and continuous (everywhere). Let \( S \) be a set (in \( \mathbb{R} \)) and let \( u \) be the supremum of \( f \) on \( S \). We are assuming that \( u \) is finite. Then there is a sequence \( \{x_n\} \) with \( x_n \in S \) so that \( f(x_n) \to u \).

**Prove or contradict 83.** Assume that \( f \) is defined and continuous (everywhere). Assume that \( S \) is bounded. Then the supremum of \( f \) on \( S \) is finite.

**Prove or contradict 84.** Assume that \( f \) is defined and continuous (everywhere). Assume that \( S \) is compact. Then \( f \) has a maximum value on \( S \).

**Prove or contradict 85.** Assume that \( f \) is defined and continuous (everywhere). Let \([a, b]\) be a bounded interval. Assume that \( f(a) < f(b) \) and fix an intermediate value \( v \), by which we mean \( f(a) < v < f(b) \). Set \( D = \{x : f(t) < v, \text{ for } a \leq t \leq x\} \). (Notice that \( x \in D \) implies that \( a \leq x \)). Show that \( D \) is nonempty and open.

**Prove or contradict 86.** Assume that \( f \) is defined and continuous (everywhere). Let \([a, b]\) be a bounded interval. Assume that \( f(a) < f(b) \) and fix an intermediate value \( v \in [f(a), f(b)] \). Then there is a point \( c \in [a, b] \) with \( f(c) = v \).

**Prove or contradict 87.** Suppose that \( f \) is continuous (on \( \mathbb{R} \)) and \( f([0, 1]) = [0, 1] \). Then \( f \) has a fixed point \( p \in [0, 1] \). That is, there is a point \( p \) in \([0, 1]\) with \( f(p) = p \). Suggestion: consider \( g(x) = f(x) - x \) or \( h(x) = x - f(x) \).

**Definition 31.** A function \( f \) defined on a domain \( D \) is **uniformly continuous** on \( D \) if given any \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for any \( x \) and \( p \) in \( D \) with \( |x - p| < \delta \), \( |f(x) - f(p)| < \varepsilon \).

**Exercise 40.** Show that \( f(x) = x^2 \) is uniformly continuous on \([0, 3]\).

**Exercise 41.** Show that \( f(x) = 1/x \) is not uniformly continuous on \((0, 1]\).

**Exercise 42.** Set \( f(x) = x \sin(1/x) \) for \( x \neq 0 \) and set \( f(0) = 0 \). Show that \( f \) is continuous at \( 0 \). Is \( f \) continuous at \( x \neq 0 \)?
Exercise 43. Set \( f(x) = x \sin(1/x) \) for \( x \neq 0 \) and set \( f(0) = 0 \). Is \( f \) uniformly continuous on \([-1,1]\)?

Prove or contradict 88. Suppose that there is a \( K \) so that for any \( x \) and \( p \) in \( D \) we have 
\[
|f(x) - f(p)| < K|x - p|
\]
Then \( f \) is uniformly continuous on \( D \).

Prove or contradict 89. Suppose that \( D \) is compact and \( f \) is continuous at each point in \( D \). Then \( f \) is uniformly continuous on \( D \).

POINT-WISE AND UNIFORM CONVERGENCE

Here we consider a sequence of real valued functions.

Definition 32. A sequence of functions \( \{f_n\} \) defined on a domain \( D \) converges to a function \( g \) on \( D \) if given any point \( x \in D \), \( \{f_n(x)\} \) converges to \( g(x) \). Notice that in this definition the point \( x \) is picked first. That is, given a point \( x \) and an \( \epsilon > 0 \), there is an \( N \) so that whenever \( n \geq N \) one also has 
\[
|f_n(x) - g(x)| < \epsilon
\]

Exercise 44. Set \( f_n(x) = x^n \) and set \( g(x) = 0 \) for \( 0 \leq x < 1 \) with \( g(1) = 1 \). Show that \( \{f_n\} \) converges to \( g \) on \([0,1]\).

Exercise 45. Set \( f_n(x) = \sin(x/n) \) and set \( g(x) = 0 \). Show that \( \{f_n\} \) converges to \( g \) (on \( \mathbb{R} \)).

Exercise 46. Set 
\[
f_n(x) = \begin{cases} 
x & 0 \leq x < \frac{1}{n} \\
2 - nx & \frac{1}{n} < x < \frac{2}{n} \\
0 & \text{otherwise}
\end{cases}
\]

Show that \( \{f_n\} \) converges to \( g(x) = 0 \) (on \( \mathbb{R} \)).

Definition 33. A sequence of functions \( \{f_n\} \) defined on a domain \( D \) converges uniformly on \( D \) if given any point any \( \epsilon > 0 \), there is an \( N \) so that whenever \( n \geq N \) one also has 
\[
|f_n(x) - g(x)| < \epsilon
\]
for all \( x \in D \).

Exercise 47. Set \( f_n(x) = x^n \). Does \( \{f_n\} \) converge uniformly on \([0,0.7]\)? on \([0,1]\)?

Exercise 48. Set \( f_n(x) = \sin(x/n) \). Does \( \{f_n\} \) converge uniformly on \( \mathbb{R} \)?

Exercise 49. Set 
\[
f_n(x) = \begin{cases} 
x & 0 \leq x < \frac{1}{n} \\
\frac{2}{n} - x & \frac{1}{n} < x < \frac{2}{n} \\
0 & \text{otherwise}
\end{cases}
\]

Does \( \{f_n\} \) converge uniformly on \( \mathbb{R} \)?

Prove or contradict 90. Fix a sequence of functions \( \{f_n\} \). Suppose that each \( f_n \) is continuous on \( D \) and that \( D \) is compact. Suppose that \( \{f_n\} \) converges uniformly on \( D \) to some function \( g \). Then \( g \) is continuous on \( D \).
RIEMANN INTEGRAL

In this section we consider integrals of functions on an interval \([a, b]\).

Definition 34. For a fixed interval \([a, b]\) a partition of \([a, b]\) is a collection of finitely many points, say \(n + 1\) points which we denote \(x_i\), with \(a = x_0 < x_1 < x_2 < \ldots < x_n = b\). The size of this partition is the length of the largest subinterval. That is size(\(\{x_i\}\)) = \(\max_{1 \leq i \leq n}(x_i - x_{i-1})\).

Definition 35. Given a function \(f(x)\) and a partition \(\{x_i\}\) of \([a, b]\), the upper sum for \(f\) on \(\{x_i\}\) is
\[
U(f, \{x_i\}) = \sum_{i=1}^{n} (x_i - x_{i-1}) \text{\text{lub}\{f(x)|x_{i-1} \leq x \leq x_i}\}}.
\]
Here \(\text{lub}\{f(x)|x_{i-1} \leq x \leq x_i\}\) is the least upper bound.

Definition 36. A function \(f\) defined on \([a, b]\) is Riemann integrable on \([a, b]\), with definite integral \(v = \int_{a}^{b} f(x)dx\), when given any \(\epsilon > 0\) there is a partition \(\{x_i\}\) of \([a, b]\), so that \(L(f, \{x_i\}) > v - \epsilon\) and \(U(f, \{x_i\}) < v + \epsilon\).

Exercise 50. Set
\[
f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}.
\]
Calculate the upper sum and lower sum for the partition of \([0, 1]\) defined by \(x_i = i/n\) for \(1 \leq i \leq n\).

Exercise 51. Set
\[
f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.
\]
Is \(f\) integrable on \([-1, 1]\)? Explain the decision.

Exercise 52. For \(f(x) = \begin{cases} x & x \text{ is not rational} \\ 0 & x \text{ is rational} \end{cases}\), and the interval \([0, 1]\) with 100 equally sized subintervals, find \(U(f, \{x_i\})\) and \(L(f, \{x_i\})\). Hint: think of the sketch of the least upper bounds and calculate the upper estimate geometrically (as an area).

Exercise 53. Set \(p_0(x) = \begin{cases} 0 & x < 1 \\ 1 & 1 \leq x \leq 1 \end{cases}\). (a) Calculate \(\int_{a}^{b} p_0(x)dx\). (b) Set \(p_1(x) = p_0(3x)\). Calculate \(\int_{0}^{1} p_1(x)dx\). (c) Set \(p_k(x) = p(3^k x)\). Calculate \(\int_{0}^{1} p_k(x)dx\). (d) Set \(s(x) = \sum_{k=0}^{\infty} p_k(x)\). Is \(s(x)\) Riemann integrable on \([0, 3]\)? Notice that we can also define
s(x) directly as the function which is 1 if \( 3^{-k} \leq x \leq 2 \cdot 3^{-k} \) for some \( k \geq 0 \) and is otherwise zero.

**Prove or contradict** 91. For \( f(x) = x \) and \([0, 1], \int_0^1 x \, dx = 1/2\).

**Prove or contradict** 92. Suppose that \( \{x_i\} \) and \( \{z_j\} \) are two partitions of \([a, b]\) so that for every \( i \) there is a \( j = j(i) \) with \( z_j = x_i \). That is, \( \{z_j\} \) is a refinement of \( \{x_i\} \). Fix a function \( f \) on \([a, b]\). Then \( U(f, \{x_i\}) \leq U(f, \{z_i\}) \). Similarly, \( L(f, \{x_i\}) \geq L(f, \{z_i\}) \).

**Prove or contradict** 93. Suppose that \( f \) is continuous (everywhere). Then \( f \) is integrable on any bounded interval \([a, b]\).

**Exercise** 54. Define \( f_n(x) \) for \( x > -1 \) by \( f_n(x) = x^n \) when \( |x| < 1 \) and \( f(x) = x^{1/n} \) for \( x \geq 1 \). What is \( g(x) \) defined by \( g(x) = \lim_{n \to \infty} f_n(x) \)?

**Prove or contradict** 94. Suppose that for each \( n \in \mathbb{N}, f_n \) is continuous. Suppose that a function \( g(x) \) exists so that for every \( x \) in some open set \( U, f_n(x) \to g(x) \) (as \( n \to \infty \)). Fix a bounded interval \([a, b] \subset U \). Then given any \( \epsilon > 0 \) there is an \( N \) so that \( |f_n(x) - g(x)| < \epsilon \) for all \( x \in [a, b] \).

**Prove or contradict** 95. Suppose that for each \( n \in \mathbb{N}, f_n \) is continuous. Suppose that a function \( g \) exists so that \( f_n(x) \to g(x) \) (as \( n \to \infty \)) for every \( x \in U \). Then \( g \) is integrable on any bounded interval \([a, b] \subset U \).

**Prove or contradict** 96. Suppose that \( \{f_n\} \) is a sequence of continuous functions. Suppose that \( \{f_n\} \) converges to \( g \) uniformly on \([a, b]\). Then \( g \) is integrable on \([a, b]\) and

\[
\int_a^b g = \lim_{n \to \infty} \int_a^b f_n.
\]

**Definition** 37. A function \( f \) defined on \([a, b]\) has **variation** \( V \) for the partition \( \{x_i\} \) of \([a, b]\), when \( V(f, \{x_i\}) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \).

**Definition** 38. A function \( f \) defined on \([a, b]\) is **of bounded variation** on \([a, b]\), when there is a bound \( B \) so that the variation of \( f \) satisfies \( V(f, \{x_i\}) \leq B \) for any partition \( \{x_i\} \) of \([a, b]\).

**Prove or contradict** 97. Suppose that \( \{x_i\} \) and \( \{z_j\} \) are two partitions of \([a, b]\) so that for every \( i \) there is a \( j = j(i) \) with \( z_j = x_i \). That is, \( \{z_j\} \) is a refinement of \( \{x_i\} \). Fix a function \( f \) on \([a, b]\). Then \( V(f, \{x_i\}) \leq V(f, \{z_i\}) \).

**Prove or contradict** 98. Suppose a function \( f \) is continuous on the bounded interval \([a, b]\). Then there is a \( K \) so that \( |f(x) - f(z)| \leq K |x - z| \) for any \( x \in [a, b] \) and \( z \in [a, b] \).

**Prove or contradict** 99. A continuous function \( f \) is of bounded variation on any bounded interval \([a, b]\).

**Prove or contradict** 100. Suppose that \( f \) is of bounded variation on \([a, b]\). Then \( f \) is integrable on \([a, b]\).
List of Some Symbols

\textbf{N} \quad \text{The natural numbers \{1, 2, 3, \ldots\}.}
\textbf{Q} \quad \text{The rational numbers.}
\textbf{R} \quad \text{The real line.}
\textbf{R}^2 \quad \text{The Euclidean plane.}
\emptyset \quad \text{The empty set.}
\forall \quad \text{For all.}
\exists \quad \text{There exists (an element).}
\setminus, A \setminus B \quad \text{Difference of sets: the elements of } A \text{ that are not elements of } B.
\overline{A} \quad \text{The closure of } A.
\in, x \in A \quad \text{Element: } x \text{ is an element of the set } A.
\notin, x \notin A \quad \text{Not an element: } x \text{ is not an element of the set } A.
\cap \quad \text{Intersection (of sets).}
\cup \quad \text{Union (of sets).}
\partial, \partial A \quad \text{The boundary of a set.}
|\cdot| \quad \text{The Euclidean distance from the origin.}
(a, b] \quad \text{The interval of real numbers, } \{x : a < x \leq b\}.

INDEX

Accumulation point of a set \quad 5
Approaches (a value or point) \quad 10
Boundary of a set \quad 6
Bounded set \quad 9
Bounded variation \quad 18
Cauchy (sequence) \quad 11
Closed set \quad 5
Closure of a set \quad 6
Collection of sets \quad 4
Compact \quad 12
Complete \quad 12
Connected set \quad 8

19
Continuous (function) 14
Converge (numbers) 10
Converges (sequence of functions) 16
Countable set 2
Cover, open 12
Decreasing (sequence) 10
Disjoint sets 7
Finite set 2
Image (of a set under a map) 15
Increasing (sequence) 10
Indexed collection of sets 4
Integral 17
Interval 9
Inverse image 15
Inverse mapping 1
Least upper bound 8
Limit value 10
Lower bound 9
Maximum 9
Minimum 9
Neighborhood of a point 5
Open set 4
One to one mapping 1
Partition of an interval 17
Separated sets 7
Sequence 9
Strictly decreasing (sequence) 10
Strictly increasing (sequence) 10
Subcover 12
Sums – upper and lower 17
Supremum 15
Surjective mapping 1
Terms (of a sequence) 9
Uniform convergence 16
Uniformly continuous 15
Upper bound 8
Variation 18