Combinatorics: Homework Set 2

November 6, 2009

1. Prove that if $|X| = n$, and if $A \subset P(X)$ is an upset (i.e. $A \in A, A \subset B$ imply $B \in A$) while $B \subset P(X)$ is a downset (i.e. $A \in B, B \subset A$ imply $B \in B$) then

\[ 2^n |A \cap B| \leq |A||B|. \]

2. For $A \subset P(X)$ write $A - A = \{ A \setminus B : A, B \in A \}$. Prove that $|A - A| \geq |A|.$

3. Show that Harper’s vertex isoperimetric inequality for the cube implies the Kruskal-Katona theorem.

4. Write down a detailed proof that the neighborhood of an initial segment of the simplicial order in the cube $Q^n$ is itself an initial segment of the simplicial order.

5. The grid $G = [k]^2$ is the graph with vertex set $[k] \times [k]$ and edges joining points $(x_1, y_1)$ and $(x_2, y_2)$ if either $x_1 = x_2$ and $|y_1 - y_2| = 1$ or $y_1 = y_2$ and $|x_1 - x_2| = 1$. State and prove a vertex isoperimetric inequality for $G$. (Hint: Use compressions.)

6. Show that every $10 \times 10$ matrix whose entries are $1, 2, \ldots, 100$ in some order has two neighboring entries (in a row or in a column) that differ by at least 10.

7. Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is finite?

8. Can a countably infinite set have an uncountable collection of nested subsets (i.e. an uncountable collection $A$ such that for $A, B \in A$, either $A \subset B$ or $B \subset A$)?

9. Construct a 2-coloring of $\mathbb{N}$ containing no infinite monochromatic arithmetic progression.

10. Construct a set $A \subset \{1, 2, \ldots, 100000\}$ of size $|A| = 1000$ which contains no 3-term arithmetic progression.