Dirichlet's Approximation Theorem

Let α be a positive real number and n a positive integer. Then Dirichlet's approximation theorem says: There is an integer k and an integer b with $0 < k < n$, for which

$$\text{(DAP)} \qquad -\frac{1}{n} < k\alpha - b < \frac{1}{n}.$$

Proof. Each of the $n+1$ numbers $a_i = i\alpha - \lfloor i\alpha \rfloor$, $i = 0, 1, 2, \ldots, n$, lies in the interval $0 \le a_i < 1$. So by the Pigeonhole Principle there is at least one (semiclosed) subinterval of the form $\left[\frac{r}{n}, \frac{r+1}{n}\right)$, $0 \le r < n$, which contains two of them, say a_m and a_j. For such an m and j either (i) $0 \le a_m - a_j < \frac{1}{n}$ or (ii) $0 \le a_j - a_m < \frac{1}{n}$ and in either situation

$$-\frac{1}{n} < a_m - a_j < \frac{1}{n};$$
$$-\frac{1}{n} < m\alpha - \lfloor m\alpha \rfloor - (j\alpha - \lfloor j\alpha \rfloor) < \frac{1}{n};$$
$$-\frac{1}{n} < (m-j)\alpha - (\lfloor m\alpha \rfloor - \lfloor j\alpha \rfloor) < \frac{1}{n};$$

setting $B = \lfloor m\alpha \rfloor - \lfloor j\alpha \rfloor$ yields both

$$-\frac{1}{n} < (m-j)\alpha - B < \frac{1}{n} \qquad \text{and} \qquad -\frac{1}{n} < (j-m)\alpha + B < \frac{1}{n}.$$

To complete the proof, if $m > j$ put $k = m - j$ and $b = B$, if $m < j$ put $k = j - m$ and $b = -B$. ∎

Now since $0 < k < n$; so $\frac{1}{kn} < \frac{1}{k^2}$, one consequence of (DAP) is that

$$\text{(1)} \qquad -\frac{1}{k^2} < -\frac{1}{nk} < \alpha - \frac{b}{k} < \frac{1}{nk} < \frac{1}{k^2}.$$

Definition 1. A "D-approximation" to α is a rational number $\frac{p}{q}$, whose denominator is a positive integer q, with

$$\text{(DAP1)} \qquad \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Dirichlet's Approximation Theorem shows that each positive real number has a D-approximation, namely the number $\frac{b}{k}$ referenced in (1).

Exercise 1. If two D-approximations both have denominator $q > 1$, then they are identical.

Exercise 2. There are at most two D-approximations with the same denominator.

Theorem 1. If α is irrational it has infinitely many D-approximations.

Proof. Suppose there is only a finite number of rational numbers $\frac{p}{q}$ satisfying (DAP1). Since $|q\alpha - p|$ is always strictly greater than zero if α is not rational, we can choose the integer n so large that

$$(\#) \qquad \frac{1}{n} < |q\alpha - p| \qquad \text{for each of these } \frac{p}{q}\text{'s.}$$

Given any n satisfying (#), Dirichlet's Approximation Theorem states that there is a positive integer $k < n$ and an integer b such that $|k\alpha - b| < \frac{1}{n}$. But then $\boxed{\left|\alpha - \frac{b}{k}\right| < \frac{1}{k^2}}$; so $\frac{b}{k}$ is a D-approximation to α and yet $\boxed{\frac{b}{k} \ne \frac{p}{q} \text{ for any of the supposedly exhaustive collection of } \frac{p}{q}\text{'s satisfying (DAP1).}}$ The contradiction shows there must be infinitely many $\frac{p}{q}$ which satisfy (DAP1). ∎

Exercise 3. Verify or prove the relation in the first boxed statement in the preceding proof.

Exercise 4. Verify or prove the relation in the second boxed statement in the preceding proof.

Theorem 2. If α is rational it has only a finite number of D-approximations.

Proof. Suppose $\frac{p}{q}$ is a D-approximation to the rational number $\frac{a}{b}$. Then if $\frac{a}{b} \ne \frac{p}{q}$,

$$\frac{1}{bq} < \frac{|aq - bp|}{bq} = \left|\frac{a}{b} - \frac{p}{q}\right| < \frac{1}{q^2} \text{ implies } q < b,$$

so there are at most b available denominators for D-approximations. It follows from exercises 1 and 2 above that the number of D-approximations to $\frac{a}{b}$ is finite. ∎