ALMOST ORTHOGONALITY AND A CLASS OF BOUNDED BILINEAR PSEUDODIFFERENTIAL OPERATORS

ÁRPÁD BÉNYI AND RODOLFO H. TORRES

Abstract. Several results and techniques that generate bilinear alternatives of a celebrated theorem of Calderón and Vaillancourt about the L^2 continuity of linear pseudodifferential operators with symbols with bounded derivatives are presented. The classes of bilinear pseudodifferential symbols considered are shown to produce continuous operators from $L^2 \times L^2$ into L^1.

1. Introduction

Bilinear and multilinear operators have received a lot of attention in recent times and are still intensively investigated. It is a natural task to try to understand which methods used to study linear pseudodifferential operators are applicable for multilinear ones and which results need to be reformulated. The purpose of this article is to explore to what extent the classical result of Calderón and Vaillancourt [2] and the techniques related to it remain valid for multilinear pseudodifferential operators.

Calderón and Vaillancourt showed in [2] that the pseudodifferential operators

$$T(f)(x) = \int_{\mathbf{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi,$$

with symbols σ satisfying estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}, \tag{1}$$

for an appropriate number of derivatives, are bounded on $L^2(\mathbf{R}^n)$. The original proof in [2] relies on a continuous version of the famous almost orthogonality lemma from the works of Cotlar [5] and Knapp and Stein [8]. We refer to Chapter VII in the book by Stein [10] for a detailed exposition of all these results.

After the work of Calderón and Vaillancourt, several other authors considered refinements of the result minimizing the number of derivatives for which (1) is assumed; see the book by Coifman and Meyer [3], the article by Cordes [4] and the references therein. Moreover, other related classes of operators for which smoothness in the x-variable in (1) is traded of by some additional size or decay estimates in the frequency variable ξ have also been considered. In particular, it is remarkable that a proof of the main result in [2] was obtained by Hwang in [7] without using the almost

Date: September 8, 2005.

1991 *Mathematics Subject Classification.* Primary 47G30, 35S05. Secondary 42B15, 42B20, 46C99.

Key words and phrases. Bilinear operators, pseudodifferential operators, almost orthogonality.

Both authors' research partially supported by the NSF under grant DMS 0070514 .

orthogonality lemma. Instead, Hwang based his elegant arguments on an elementary identity involving the Wigner transform, see Lemma 2 below. As a corollary he also obtained the L^2-boundedness of pseudodifferential operators with symbols satisfying the estimates

$$\|\partial_\xi^\beta \sigma(x,\cdot)\|_{L^2} \leq C_\beta, \tag{2}$$

uniformly in x, and where $\beta = (\beta_1, \dots, \beta_n)$ with each $\beta_j = 0$ or 1. This result is Corollary 2.1 in [7].

We investigate the above results and approaches for bilinear pseudodifferential operators of the form

$$T(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta.$$

First, we observe an extension of the almost orthogonality lemma to the bilinear setting, Lemma 1 below, which is of interest in its own. We show in Proposition 1 that the analogous estimates to (1) do not produce in general bounded operators from $L^2 \times L^2$ into L^1. Nevertheless, we illustrate a possible use of Lemma 1 in Theorem 1, obtaining a bilinear substitute of the results in [2] by imposing some extra size estimates on the symbols in the frequency variables. As in the linear case it is possible also to obtain this type of results without using explicit almost orthogonality arguments. We show this with the proof given in Theorem 2, where we make use of the identity exploited by Hwang in [7] that we alluded to before.

Acknowledgement. We are indebted to the anonymous referee for pointing out a reference and many valuable suggestions that simplified our presentation.

2. A bilinear almost orthogonality lemma

Let $B_i, i = 1,2,3$, be three normed spaces and B_i^* their dual spaces. A bilinear operator $T : B_1 \times B_2 \to B_3$ is linear in every entry and consequently has two formal transposes $T^{*1} : B_3^* \times B_2 \to B_1^*$ and $T^{*2} : B_1 \times B_3^* \to B_2^*$ defined via

$$\langle b_3^*, T(b_1,b_2)\rangle_{B_3^*,B_3} = \langle T^{*1}(b_3^*,b_2), b_1\rangle_{B_1^*,B_1} = \langle T^{*2}(b_1,b_3^*), b_2\rangle_{B_2^*,B_2},$$

for all $b_1 \in B_1, b_2 \in B_2, b_3^* \in B_3^*$. Here $\langle\cdot,\cdot\rangle_{B^*,B}$ denotes the dual pairing. We will also use the notation $\|T\| = \sup \|T(f,g)\|_{B_3}$, where the supremum is taken over all $f \in B_1$ and $g \in B_2$ with $\|f\|_{B_1} = \|g\|_{B_2} = 1$.

Let H denote a (complex) Hilbert space endowed with an inner product $\langle\cdot|\cdot\rangle$ and let V be a (complex) normed space of functions closed under conjugation, i.e., $f \in V$ implies $\bar{f} \in V$ and $\|\bar{f}\|_V = \|f\|_V$. As usual, we identify H^* with H via the anti-isomorphism $H^* \ni f^* \mapsto J_H f^* \in H$, $\langle f^*, f\rangle_{H^*,H} = \langle f|J_H f^*\rangle$. Hence, if $T : V \times H \to H$ is a bilinear operator, one can view $T^{*2} : V \times H^* \to H^*$ as a bilinear operator from $V \times H$ into H given by

$$V \times H \ni (f,g) \mapsto J_H T^{*2}(\bar{f}, J_H^{-1} g) \in H.$$

With some abuse, we will still call this operator T^{*2}. With this identification, $\langle T^{*2}(f,g)|h\rangle = \langle g|T(\bar{f},h)\rangle$, for all $f \in V$ and $g,h \in H$. It follows that if T_f :

$H \to H$ is given by $T_f(g) = T(f,g)$, then the Hilbert space adjoint of T_f is given by $(T_f)^* = (T^{*2})_{\bar{f}}$. Indeed, for any $g, h \in H$,

$$\langle T_f^*(g)|h\rangle = \langle g|T_f(h)\rangle = \langle g|T(f,h)\rangle = \langle T^{*2}(\bar{f},g)|h\rangle = \langle (T^{*2})_{\bar{f}}(g)|h\rangle.$$

An m-linear form of the following lemma was first obtained in [1]. The version we present here will suffice for the purposes of this article.

Lemma 1. *Let H be a Hilbert space and V a normed space of functions closed under conjugation. If $T_j : V \times H \to H, j \in \mathbf{Z}$, is a sequence of bounded bilinear operators and $\{a(j)\}_{j\in\mathbf{Z}}$ is a sequence of positive real numbers such that*

$$\|T_i(f, T_j^{*2}(\bar{f},g))\|_H + \|T_i^{*2}(\bar{f}, T_j(f,g))\|_H \le a(i-j), \tag{3}$$

for all $f \in V, g \in H, \|f\|_V = \|g\|_H = 1$, and for all $i, j \in \mathbf{Z}$, then

$$\|\sum_{j=n}^{m} T_j\| \le \sum_{i=-\infty}^{\infty} \sqrt{a(i)},\ \ n,m \in \mathbf{Z}, n \le m.$$

Proof. The proof follows from the linear case by "freezing" one function. In fact, since $T_j : V \times H \to H$ is a bounded bilinear operator, if we freeze $f \in V$, $\|f\|_V = 1$, it follows that $T_{jf} : H \to H, T_{jf}(g) = T_j(f,g)$, is a bounded linear operator. As noted before, the Hilbert space adjoint of the linear operator T_{jf} is given by $T_{jf}^* = (T_j^{*2})_{\bar{f}}$. Hence, we can now rewrite the condition (3) of the hypothesis as

$$\|T_{if}T_{jf}^*(g)\|_H + \|T_{if}^*T_{jf}(g)\|_H \le a(i-j),\ \text{ for all }\ g \in H, \|g\|_H = 1, i,j \in \mathbf{Z}. \tag{4}$$

If we take the supremum on the inequality (4) over all $g \in H, \|g\|_H = 1$, we get

$$\|T_{if}T_{jf}^*\| + \|T_{if}^*T_{jf}\| \le a(i-j),\ \ i,j \in \mathbf{Z}.$$

These are the nowadays well-known conditions for the linear case (see e.g. [10] p. 279) applied to the family of bounded linear operators $T_{jf} : H \to H$ and, hence,

$$\|\sum_{j=n}^{m} T_{jf}\| \le \sum_{i=-\infty}^{\infty} \sqrt{a(i)},\ \ n,m \in \mathbf{Z}, n \le m.$$

If we now take the supremum over all $f \in V, \|f\| = 1$, we get

$$\sup\{\|\sum_{j=n}^{m} T_j(f,g)\|_H : \|f\|_V = \|g\|_H = 1\} \le \sum_{i=-\infty}^{\infty} \sqrt{a(i)},\ \ n,m \in \mathbf{Z}, n \le m,$$

as we wanted to prove. □

Remark 1. The result in the above lemma still holds if we replace the summation over the integers with a summation over $\mathbf{Z}^n$. Similarly, the same method of freezing one function can be used to obtain continuous versions analogous to the linear one in [2].

3. Some classes of bilinear pseudodifferential operators

We study bilinear pseudodifferential operators of the form

$$T_\sigma(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

which are á priori defined on $\mathcal{S}\times\mathcal{S}$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing, C^∞ functions in $\mathbf{R}^n$, and where we choose the Fourier transform to be given by $\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-ix\cdot\xi}\,dx$. When the symbol will be implicitly understood from the context, we will simply write T instead of T_σ.

Consider first symbols satisfying the differential inequalities

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}, \tag{5}$$

for all $(x,\xi,\eta)\in\mathbf{R}^{3n}$ and all multi-indices α, β, and γ. Obviously, for $\sigma=1$ we obtain the multiplication of two functions which is a bounded operation from $L^2\times L^2$ into L^1 or from $L^p\times L^q$ into L^r for $1/p+1/q=1/r$. More generally, the same boundedness properties hold if $\sigma(\xi,\eta)=\sigma_1(\xi)\sigma_2(\eta)$ where σ_1 gives rise to a Fourier multiplier in L^p and σ_2 to one in L^q. Nevertheless, not all the operators with symbols satisfying (5) enjoy this boundedness property even if their symbols are x-independent. In fact we have the following simple proposition.

Proposition 1. *There exist operators with x-independent symbols $\sigma(\xi,\eta)$ satisfying (5) which are not bounded from $L^p\times L^q$ into L^r for $1/p+1/q=1/r$, $1\leq p,q,r<\infty$.*

Proof. If $p\neq 2$ simply take a symbol of the form $\sigma(\xi,\eta)=\sigma_1(\xi)$ where σ_1 satisfies (5) but is not a multiplier in L^p. Similarly if $q\neq 2$.

Let then $p=q=2$. Suppose by contradiction that every symbol $\sigma(\xi,\eta)$ satisfying (5) defines a bounded operator T_σ from $L^2\times L^2$ into L^1, and consider a symbol of the form $\sigma(\xi,\eta)=\rho(-\xi-\eta)$. By duality, then $(T_\sigma)^{*1}: L^\infty\times L^2\to L^2$. Let L^∞_c be the space of L^∞ functions with compact support. Although multipliers are not á priori defined on a dense subspace of L^∞ it is easy to see (we spare the reader the elementary computations that show this fact) that the above operator defined by duality agrees on $L^\infty_c\times L^2\to L^2$ with the bilinear multiplier with symbol given by $\sigma^{*1}(\xi,\eta)=\sigma(-\xi-\eta,\eta)=\rho(\xi)$, which also satisfies (5). It would follow that every operator with a symbol of the form $\sigma(\xi,\eta)=\sigma_1(\xi)$ satisfying (5) defines a bounded bilinear operator from $L^2\times L^2$ into L^1 and from $L^\infty_c\times L^2\to L^2$. But then it would also map (by linear interpolation freezing the second function) $L^p\times L^2$ into $L^{2p/(2+p)}$ for any $2<p<\infty$, and we already saw that this is not possible.

Explicit symbols $\sigma(\xi)$ satisfying (5) which are not Fourier multiplier in L^p, $p\neq 2$ are for example given in [10], p. 322 and the work of Wainger [11]. □

Next, we can obtain a positive result by adding some size conditions on the frequency variables. Consider symbols satisfying the following inequalities

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}, \tag{6}$$

$$\sup_x \int \left(\int |\partial_\eta^\alpha \sigma(x,\xi,\eta)|^2\,d\xi\right)^{1/2} d\eta \leq C_\alpha \tag{7}$$

and

$$\sup_x \int \left(\int |\partial_\xi^\alpha \sigma(x,\xi,\eta)|^2 \, d\eta \right)^{1/2} d\xi \le C_\alpha, \tag{8}$$

for all multi-indices α, β, γ.

Theorem 1. *Let T be a pseudodifferential operator whose symbol satisfies (6)-(8). Then T can be extended as a bounded operator from $L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$.*

Proof. Note that $T(f,g) = S(\widehat{f}, \widehat{g})$, where

$$S(f,g)(x) = \int\int \sigma(x,\xi,\eta) f(\xi) g(\eta) e^{ix\cdot(\xi+\eta)} d\xi d\eta. \tag{9}$$

Thus, by Plancherel's theorem, the boundedness of $S : L^2 \times L^2 \to L^1$ is equivalent to the boundedness of T. Moreover, by duality, it is enough to show that the operator

$$R(f,g)(x) = \int\int \sigma(\xi,x,\eta) f(\xi) g(\eta) e^{i\xi\cdot(x+\eta)} d\xi d\eta. \tag{10}$$

is bounded from $L^\infty \times L^2$ into L^2. We will assume first that σ has compact support, but the estimates we will obtain will not depend on the support of the symbol.

In order to apply Lemma 1, we compute the $*2$-adjoint of R in the Hilbert space sense explained in the previous section and obtain

$$R^{*2}(f,g)(x) = \int\int \overline{\sigma(\xi,\eta,x)} f(\xi) g(\eta) e^{-i\xi\cdot(\eta+x)} d\xi d\eta. \tag{11}$$

Choose a smooth, nonnegative function ϕ such that

$$\operatorname{supp} \phi \subset \{\xi = (\xi_1, \xi_2, ..., \xi_n) : |\xi_j| \le 1, j = 1, 2, ..., n\},$$

and

$$\sum_{i \in \mathbf{Z}^n} \phi(\xi - i) = 1.$$

Set

$$\sigma_i(\xi,x,\eta) = \sigma(\xi,x,\eta)\phi(\xi - i),$$

and write R_i for the operator (10) with $\sigma(\xi,x,\eta)$ replaced by $\sigma_i(\xi,x,\eta)$, so that

$$R = \sum_{i \in \mathbf{Z}^n} R_i.$$

Straightforward computations show that for $f \in L^\infty$ and $g \in \mathcal{S}$,

$$R_i(f, R_j^{*2}(\bar{f}, g))(x) = \int \sigma_{ij}(x,x') g(x') dx',$$

where

$$\sigma_{ij}(x,x') = \int\int\int \sigma_i(\xi,x,\eta) \overline{\sigma_j(\xi',x',\eta)} f(\xi) \overline{f(\xi')} e^{i\xi\cdot(x+\eta)} e^{-i\xi'\cdot(x'+\eta)} d\xi d\xi' d\eta. \tag{12}$$

Similarly, we have

$$R_i^{*2}(\bar{f}, R_j(f,g))(x) = \int \tilde{\sigma}_{ij}(x,x') g(x') dx',$$

where

$$\tilde{\sigma}_{ij}(x,x') = \int\int\int \overline{\sigma_i(u,v,x)}\sigma_j(u',v,x')\overline{f(u)}f(u')e^{iu'\cdot(v+x')}e^{-iu\cdot(v+x)}dudu'dv. \tag{13}$$

Set $f_i(\xi) = f(\xi)\phi(\xi - i)$. We have

$$\begin{aligned}
&(2\pi)^n R_i(f, R_j^{*2}(\bar{f}, g))(x) \\
&= (2\pi)^n \int\int\int\int \sigma(\xi,x,\eta)\overline{\sigma(\xi',x',\eta)}f_i(\xi)\overline{f_j(\xi')}g(x')e^{i\xi\cdot(x+\eta)}e^{-i\xi'\cdot(x'+\eta)}d\xi d\xi' dx' d\eta \\
&= \int\int\int\int\int \widehat{\sigma}^2(\xi,y,\eta)\overline{\sigma(\xi',x',\eta)}f_i(\xi)\overline{f_j(\xi')}g(x')e^{iy\cdot x}e^{i\xi\cdot(x+\eta)-i\xi'\cdot(x'+\eta)}dyd\xi d\xi' dx' d\eta \\
&= \int F_{ij}(x,\xi)e^{i\xi\cdot x}\,d\xi.
\end{aligned}$$

Here $\widehat{\sigma}^2$ denotes the Fourier transform of σ in the second variable. Let $N = n+1$. Since F_{ij} has compact support in ξ of fixed size, it will be enough to prove that

$$\|F_{ij}(\cdot,\xi)\|_{L^2} \le C_N(1+|i-j|)^{-2N}\|g\|_{L^2}\|f\|_{L^\infty}^2.$$

Moreover, writing

$$\begin{aligned}
&F_{ij}(x,\xi) \\
&= \int\int\int\int \widehat{\sigma}^2(\xi,y,\eta)\overline{\sigma(\xi',x',\eta)}g(x')\overline{f_j(\xi')}f_i(\xi)e^{-i\eta\cdot(\xi'-\xi)}e^{-i\xi'\cdot x'}e^{iy\cdot x}dyd\xi' dx' d\eta \\
&= \int G_{ij}(y,\xi)e^{iy\cdot x}dy,
\end{aligned}$$

and using Plancherel's theorem, it is enough to show that

$$\|G_{ij}(\cdot,\xi)\|_{L^2} \le C_N(1+|i-j|)^{-2N}\|g\|_{L^2}\|f\|_{L^\infty}^2, \tag{14}$$

where

$$G_{ij}(y,\xi) = \int\int\int \widehat{\sigma}^2(\xi,y,\eta)\overline{\sigma(\xi',x',\eta)}g(x')\overline{f_j(\xi')}f_i(\xi)e^{i\eta\cdot(\xi-\xi')}e^{-i\xi'\cdot x'}d\xi' dx' d\eta.$$

If we integrate by parts with respect to η we obtain

$$\begin{aligned}
&G_{ij}(y,\xi) = \\
&\sum_{|\alpha|+|\beta|\le 2N} c_{\alpha\beta}\int\int\int \frac{\partial_\eta^\alpha\widehat{\sigma}^2(\xi,y,\eta)\partial_\eta^\beta\overline{\sigma(\xi',x',\eta)}}{(1+|\xi-\xi'|^2)^N}g(x')\overline{f_j(\xi')}f_i(\xi)e^{i\eta\cdot(\xi-\xi')}e^{-i\xi'\cdot x'}d\xi' dx' d\eta.
\end{aligned}$$

Since the integration in ξ' take place also over a compact set of fixed size, we get

$$\begin{aligned}
&|G_{ij}(y,\xi)| \\
&\le C_N \int |\partial_\eta^\alpha \widehat{\sigma}^2(\xi,y,\eta)| \int \left| \frac{e^{i\eta\cdot(\xi-\xi')}\overline{f_j(\xi')}f_i(\xi)}{(1+|\xi-\xi'|^2)^N} \int \partial_\eta^\beta \overline{\sigma(\xi',x',\eta)} g(x') e^{-i\xi'\cdot x'}dx' \right| d\xi' d\eta \\
&= C_N \int |\partial_\eta^\alpha \widehat{\sigma}^2(\xi,y,\eta)| \int |\frac{e^{i\eta\cdot(\xi-\xi')}}{(1+|\xi-\xi'|^2)^N}\overline{f_j(\xi')}f_i(\xi) H_{\beta\eta}(\widehat{g})(\xi')| d\xi' d\eta \\
&\le C_N \int |\partial_\eta^\alpha \widehat{\sigma}^2(\xi,y,\eta)| (1+|i-j|)^{-2N} \|f\|_{L^\infty}^2 \|H_{\beta\eta}(\widehat{g})\|_{L^2} d\eta.
\end{aligned}$$

Note that, because of (6), $H_{\beta\eta}$ is a (linear) pseudodifferential operator,

$$H_{\beta\eta}(h)(x) = \int a_{\beta\eta}(x,\xi)\widehat{h}(\xi)e^{ix\cdot\xi}\,d\xi,$$

with symbol $a_{\beta\eta}(x,\xi) = \partial_\eta^\beta \overline{\sigma(x,-\xi,\eta)}$ satisfying (1) uniformly in η. By the results of Calderón and Vaillancourt

$$|G_{ij}(y,\xi)| \le C_N(1+|i-j|)^{-2N}\|f\|_{L^\infty}^2 \|g\|_{L^2} \int |\partial_\eta^\alpha \widehat{\sigma}^2(\xi,y,\eta)| d\eta.$$

The last inequality implies that

$$\begin{aligned}
\|G(\cdot,\xi)\|_{L^2} &\le C_N(1+|i-j|)^{-2N}\|f\|_{L^\infty}^2 \|g\|_{L^2} \| \int |\partial_\eta^\alpha \widehat{\sigma}^2(\xi,\cdot,\eta)| d\eta\|_{L^2} \\
&\le C_N(1+|i-j|)^{-2N}\|f\|_{L^\infty}^2 \|g\|_{L^2} \int \|\partial_\eta^\alpha \widehat{\sigma}^2(\xi,\cdot,\eta)\|_{L^2} d\eta \\
&\le C_N(1+|i-j|)^{-2N}\|f\|_{L^\infty}^2 \|g\|_{L^2} \int \left(\int |\partial_\eta^\alpha \sigma(\xi,y,\eta)|)^2 dy \right)^{1/2} d\eta \\
&\le C_N(1+|i-j|)^{-2N}\|f\|_{L^\infty}^2 \|g\|_{L^2}.
\end{aligned}$$

Here we used successively Minkowski's inequality, Plancherel's theorem and the condition (7). This gives the almost orthogonality for $R_i(f, R_j^{*2}(\bar{f}, g))$. The same approach, using now (8), proves the almost orthogonality for $R_i^{*2}(\bar{f}, R_j(f,g))$.

We now remove the condition on the support of σ by a standard argument. Fix a C^∞ function u with compact support in $\mathbf{R}^{3n}$ and such that $u(0,0,0) = 1$ and set $\sigma_\epsilon(x,\xi,\eta) = \sigma(x,\xi,\eta)u(\epsilon x, \epsilon\xi, \epsilon\eta)$. Notice that if the symbol σ satisfies (6)-(8), then the symbols σ_ϵ satisfy the same inequalities uniformly in ϵ, for $0 < \epsilon \le 1$. The symbols σ_ϵ have compact support and $S_{\sigma_\epsilon}(f,g) \to S_\sigma(f,g)$ a.e. as $\epsilon \to 0$ for all $f, g \in \mathcal{S}$. Since the operators S_{σ_ϵ} are bounded from $L^2 \times L^2$ into L^1 with bounds independent of ϵ, by letting $\epsilon \to 0$ we obtain the boundedness of the operator S_σ. □

Remark 2. One can replace the conditions in (6) by the conditions

$$\|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\eta)\|_{L^2} \le C_{\alpha\beta}, \tag{15}$$

uniformly in x and η, and

$$\|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\xi,\cdot)\|_{L^2} \le C_{\alpha\beta}, \tag{16}$$

uniformly in x and ξ together with (7) and (8) if one uses in the above proof Corollary 2.1 in [7] instead of the original result of Calderón-Vaillancourt. The two sets of conditions are not comparable. Certainly, in either case one does not need to assume the estimates for all derivatives but only an appropriate number of them. We do not compute the smallest such number because we will present, by further using some ideas in [7], another variant without smoothness in the x-variable .

We recall the following identity; Lemma 3.1 in [7].

Lemma 2. *For f and φ in $L^2(\mathbf{R}^n)$, define*

$$F(x,\xi) = \int_{\mathbf{R}^n} e^{-iy\cdot\xi}\varphi(x-y)f(y)\,dy. \tag{17}$$

Then,

$$\|F\|_{L^2(\mathbf{R}^n\times\mathbf{R}^n)} = (2\pi)^{n/2}\|\varphi\|_{L^2(\mathbf{R}^n)}\|f\|_{L^2(\mathbf{R}^n)}.$$

The proof of the lemma is straightforward, see [7]. When φ has compact support, the integral in (17) is sometimes called the short time Fourier transform of f with window $\overline{\varphi}$. The integral is very similar to the Wigner transform of φ and f,

$$W(f,\varphi)(x,\xi) = \int_{\mathbf{R}^n} e^{-iy\cdot\xi}\varphi(x+y/2)\overline{f(x-y/2)}\,dy.$$

For more on the use of the Wigner transform to study pseudodifferential operators see e.g. Chapter 1 in the book by Folland [6].

Theorem 2. *Let T be a pseudodifferential operator whose symbol satisfies the inequalities*

$$\sup_x \|\partial_{\xi_j}^{\alpha_j}\partial_{\eta_k}^{\beta_k}\sigma(x,\cdot,\cdot)\|_{L^2(\mathbf{R}^n\times\mathbf{R}^n)} \le C \tag{18}$$

for all $j,k = 1,\ldots,n$, and $\alpha_j,\beta_k = 0$ or 1. Then, T can be extended as a bounded operator from $L^2(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$.

Proof. Without loss of generality, we may assume that the symbol has compact support. This will ensure the absolute convergence of all the integrals to be considered. The estimates to be obtained, however, will not depend on the support of the symbol and the same limiting argument used in Theorem 1 can be applied to prove the general case.

Let f, g be functions in $\mathcal{S}(\mathbf{R}^n)$. By first inverting the Fourier transform, then integrating by parts and finally using Cauchy-Schwarz in ξ and η and (18), we compute

$$\begin{aligned}
&\|T(f,g)\|_{L^1(\mathbf{R}^n)} \\
&= \int \left| \int\int \sigma(x,\xi,\eta) e^{ix\cdot(\xi+\eta)} \widehat{f}(\xi)\widehat{g}(\eta)\, d\xi d\eta \right| dx \\
&= \int \left| \int\int \sigma(x,\xi,\eta) \left(\int e^{i\xi\cdot(x-y)} f(y)dy\right) \left(\int e^{i\eta\cdot(x-z)} g(z)dz\right) d\xi d\eta \right| dx \\
&= \int \left| \int\int \rho(x,\xi,\eta) H(x,\xi,\eta)\, d\xi d\eta \right| dx \\
&\le \sup_x \|\rho(x,\cdot,\cdot)\|_{L^2(\mathbf{R}^n\times\mathbf{R}^n)} \int \left(\int\int |H(x,\xi,\eta)|^2\, d\xi d\eta\right)^{1/2} dx \\
&\le C \int \left(\int\int |H(x,\xi,\eta)|^2\, d\xi d\eta\right)^{1/2} dx,
\end{aligned}$$

where

$$\rho(x,\xi,\eta) = \left(\prod_{j=1}^n (1-\partial_{\xi_j}) \prod_{k=1}^n (1-\partial_{\eta_k})\right) \sigma(x,\xi,\eta),$$

$$H(x,\xi,\eta) = \left(\int e^{i\xi\cdot(x-y)} \varphi(x-y) f(y)dy\right) \left(\int e^{i\eta\cdot(x-z)} \varphi(x-z) g(z)dz\right),$$

and

$$\varphi(x) = \prod_{j=1}^n \frac{1}{1+ix_j}.$$

Since the last integral in ξ and η in the previous computations splits, we can use Cauchy-Schwarz in x and Lemma 2 to obtain

$$\begin{aligned}
&\|T(f,g)\|_{L^1(\mathbf{R}^n)} \\
&\le C \int \left(\int \left| \int e^{-iy\xi} \varphi(x-y) f(y) dy \right|^2 d\xi\right)^{1/2} \left(\int \left| \int e^{-iz\eta} \varphi(x-z) g(z) dz \right|^2 d\eta\right)^{1/2} dx \\
&\le C \|\varphi\|^2_{L^2(\mathbf{R}^n)} \|f\|_{L^2(\mathbf{R}^n)} \|g\|_{L^2(\mathbf{R}^n)},
\end{aligned}$$

which implies the theorem. □

Remark 3. We comment some more on the hypotheses in Theorem 1 and Theorem 2. If one assumes estimates on the symbol similar to (7) and (8) but including mixed derivatives, then the hypotheses of Theorem 2 are easily verified. To see this, we write the inequalities in terms of mixed Lebesgue norms $L^p_\xi(L^q_\eta)$ and use interpolation. For example, change the conditions in Theorem 1 to

$$\sup_x \|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\cdot)\|_{L^\infty(L^\infty)} \le C \tag{19}$$

$$\sup_x \|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\cdot)\|_{L^1_\xi(L^2_\eta)} \le C \tag{20}$$

and

$$\sup_x \|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\cdot)\|_{L^1_\eta(L^2_\xi)} \le C. \tag{21}$$

Then, from (19) and (20) we get, in particular,

$$\sup_x \|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\cdot)\|_{L^2_\xi(L^4_\eta)} \le C. \tag{22}$$

From (21) we also get

$$\sup_x \|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\cdot)\|_{L^2_\xi(L^1_\eta)} \le C, \tag{23}$$

and then from (22) and (23) we get

$$\sup_x \|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\cdot)\|_{L^2(\mathbf{R}^n\times\mathbf{R}^n)} = \sup_x \|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\cdot,\cdot)\|_{L^2_\xi(L^2_\eta)} \le C. \tag{24}$$

If one only assumes (7) and (8) but with sufficient number of derivatives, it may still be possible to obtain (18) through more elaborate arguments using complex interpolation in mixed Lebesgue spaces. Such interpolation arguments, however, are beyond the main focus of this paper. For complex interpolation involving derivatives estimates in mixed lebesgue spaces, we refer the interested reader to [9].

The conditions in Theorem 2 are trivially satisfied if

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\xi,\eta)| \le C(1+|\xi|+|\eta|)^{-n-\epsilon}, \tag{25}$$

but the ones in Theorem 1 do not necessarily follow from (25), even if we add derivatives in x. The symbols satisfying (25) are the bilinear analog of the linear pseudodifferential operators with "rough" symbols σ such that

$$|\partial_\xi^\alpha \sigma(x,\xi)| \le C_\alpha(1+|\xi|)^{-n/2-\epsilon}. \tag{26}$$

The operators with symbols satisfying (25) are not pseudo-local in the sense that their kernels are not smooth away from the diagonal and they do not have fast decay at infinity. If instead of the conditions (25) we require the symbols to satisfy inequalities of the form

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\beta\gamma}(1+|\xi|+|\eta|)^{-|\beta|-|\gamma|-\epsilon}, \tag{27}$$

or

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\beta\gamma}(1+|\xi|+|\eta|)^{-2n-\epsilon}, \tag{28}$$

then the corresponding pseudodifferential operators will be easily bounded. Indeed, in such a case the distribution kernels of the bilinear pseudodifferential operators satisfy the inequalities

$$|K(x,y,z)| \le C(|x-y|+|x-z|)^{-2n+\epsilon}, \tag{29}$$

$$|K(x,y,z)| \le C(|x-y|+|x-z|)^{-2n-\epsilon}, \text{ for } |x-y|+|x-z|\ge 1. \tag{30}$$

Elementary computations using Hölder's inequality and the Hardy-Littlewood maximal function give that the operators which such estimates in their kernels are bounded from $L^p \times L^q$ into L^r, for $1/p+1/q=1/r$ and $r\ge 1$. The estimates (29) and (30) on the distribution kernel cannot be obtained if the symbol satisfies (25) or, in the linear case, (26).

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Árpád Bényi, Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA,

E-mail address: benyi@math.ukans.edu

Rodolfo H. Torres, Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA,

E-mail address: torres@math.ukans.edu