GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.

On in class exams I assign four problems. Each is worth 25 points.

I try to assign problems from different topics that we covered.

Below are several problems to help you get used to my style of exam questions.

1. Let \( v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, v_5 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \) be given vectors in \( \mathbb{R}^4 \).

(a) Row reduce the \( 4 \times 5 \) matrix whose columns are the given vectors. Use the reduced row echelon form to answer the questions below.

(b) Are vectors \( v_1, v_2, v_3, v_4, v_5 \) linearly independent? Give justification for your answer based on the definition. Be specific!

(c) Find a basis for \( \text{span}\{v_1, v_2, v_3, v_4, v_5\} \). What is the dimension of this space? Do vectors \( v_1, v_2, v_3, v_4, v_5 \) span \( \mathbb{R}^4 \)? Again, be specific!

(d) What is the dimension of \( V = \text{span}\{v_1, v_2, v_3\} \)? What is the dimension of \( W = \text{span}\{v_4, v_5\} \)? Find a nonzero vector \( u \) which belongs to both subspaces \( V \) and \( W \). Be specific! Give \( u \) as a linear combination of \( v_1, v_2, v_3 \) and of \( v_4, v_5 \). A complete answer to (1b) can help here.

2. Let \( A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 7 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & -2 & -1 \\ -2 & -1 & 0 \\ -7 & 4 & 3 \end{bmatrix} \). Determine whether the matrices \( A \) and \( B \) have the same column space. (Important Note: If you claim that \( A \) and \( B \) have the same column space justify your claim by calculations. If you claim that the matrices do not have the same column space give a specific vector which is in one column space but not in the other.)

3. (a) Let \( v_1, \ldots, v_m \) be vectors in a vector space \( V \). State the definition of linear independence for the vectors \( v_1, \ldots, v_m \).

The rest of this problem is about the space \( \mathbb{P}_2 \) of all polynomials of degree at most 2.

(b) Consider the polynomials \( q_1(t) = t, q_2(t) = 1 - t^2, q_3(t) = 1 + t^2 \) in \( \mathbb{P}_2 \). Are these polynomials linearly independent in \( \mathbb{P}_2 \)? Why or why not?

(c) Let \( H \) be the set of all polynomials \( p \) in \( \mathbb{P}_2 \) such that \( p(-1) = p(1) \). Show that \( H \) is a subspace of \( \mathbb{P}_2 \).

(d) Find a basis for \( H \).

4. Suppose that \( A \) is \( 12 \times 17 \) matrix and let \( S : \mathbb{R}^{17} \to \mathbb{R}^{12} \) be a linear transformation given by \( S(x) = Ax \).

Suppose that \( B \) is \( 17 \times 12 \) matrix and let \( T : \mathbb{R}^{12} \to \mathbb{R}^{17} \) be a linear transformation given by \( T(x) = Bx \).

(a) Can \( S \) be one-to-one? Why or why not? Can \( S \) be onto? Why or why not? Is there a connection between \( S \) being one-to-one or onto and \( \text{Nul} \ A \)?
(b) Can $T$ be one-to-one? Why or why not? Can $T$ be onto? Why or why not? Is there a connection between $T$ being one-to-one or onto and $\text{Nul } B$?

(c) If $b \in \mathbb{R}^{12}$ is such that $Ax = b$ has no solution, what can you conclude about the dimension of $\text{Nul } A$? (Give your answer in the form: $?? \leq \dim \text{Nul } A \leq ??$, where $??$ stand for a specific integer.)

(d) If $b \in \mathbb{R}^{17}$ is such that $Bx = b$ has no solution, what can you conclude about the dimension of $\text{Nul } B$? (Give your answer in the form: $?? \leq \dim \text{Nul } B \leq ??$, where $??$ stand for a specific integer.)

(e) Which of the matrices $AB$, $BA$ is defined? Which of these matrices could be invertible? Which of these matrices is never invertible? Explain your answers.

5. In this problem we consider the vector space $\mathbb{P}_2$ of all polynomials of degree at most 2. Recall that the standard basis for $\mathbb{P}_2$ is $\mathcal{S} = \{1, t, t^2\}$.

(a) Consider the polynomials $q_1(t) = 1 - t^2, q_2(t) = t, q_3(t) = 1 + t^2$ in $\mathbb{P}_2$. Prove that $\mathcal{B} = \{q_1, q_2, q_3\}$ is a basis for $\mathbb{P}_2$.

(b) Find $P_{\mathcal{B} \leftarrow \mathcal{S}}$.

(c) Find the basis $\mathcal{C}$ for $\mathbb{P}_2$ if $P_{\mathcal{C} \leftarrow \mathcal{B}} = A$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

6. Consider the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}$.

(a) Find all eigenvalues of $A$.

(b) For each eigenvalue find a corresponding eigenvector.

7. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \\ -4 & 4 & -5 \end{bmatrix}$. The eigenvalues of this matrix are $-1$ and $-2$. Find all eigenvectors corresponding to the eigenvalue $-1$.

8. Below I give two $3 \times 5$ matrices $A$ and $B$ together with their reduced row echelon forms:

\[
A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 1 & 2 & 2 & 4 & 5 \\ 2 & 4 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 4 & 3 & 1 & 2 & 1 \\ 5 & 4 & 2 & 2 & 1 \\ 7 & 5 & 1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 2 & 1 \\ 0 & 1 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

Denote the columns of $A$ by $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$. Notice that the columns of the matrix $B$ are the columns of the matrix $A$ in reversed order. Therefore $\text{Col } A = \text{Col } B$.

(a) i. Which basis for $\text{Col } A$ is determined by the given RREF of $A$? Call this basis $\mathcal{A}$.

ii. Which basis for $\text{Col } B$ is determined by the given RREF of $B$? Call this basis $\mathcal{B}$.
(b) Based on the given RREFs for $A$ and $B$ find $P_{A \leftarrow B}$ and $P_{B \leftarrow A}$.

9. By $\mathbb{R}^{2 \times 2}$ we denote the vector space of all $2 \times 2$ matrices. Let $A$ be a matrix in $\mathbb{R}^{2 \times 2}$. Decide which of the following are subspaces and justify your answer.

(a) $\mathcal{H} = \{ B \in \mathbb{R}^{2 \times 2} : AB = BA \}$
(b) $\mathcal{H} = \{ B \in \mathbb{R}^{2 \times 2} : BA = 0 \}$
(c) $\mathcal{H} = \{ B \in \mathbb{R}^{2 \times 2} : BB^\top = 0 \}$
(d) $\mathcal{H} = \{ B \in \mathbb{R}^{2 \times 2} : \det(B) = 0 \}$

10. Consider the following matrices

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 1 \\
0 & 1 & 0 \\
2 & 1 & 3
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
1 & 1 & 0 \\
2 & 2 & 1 \\
3 & 2 & 3 \\
4 & 3 & 6
\end{bmatrix}
\]

Determine all vectors which belong to both $\text{Col}_A$ and $\text{Col}_B$. 