CHAPTER 3

Continuous functions

In this chapter $I$ will always denote a non-empty subset of $\mathbb{R}$.

3.1. The $\varepsilon$-$\delta$ definition of a continuous function

Definition 3.1.1. A function $f : I \rightarrow \mathbb{R}$ is continuous at a point $x_0 \in I$ if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that

$$x \in (x_0 - \delta, x_0 + \delta) \cap I \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$ (3.1.1)

The function $f$ is continuous on $I$ if it is continuous at each point of $I$.

Note that the implication in (3.1.1) can be restated as

$$x \in I \text{ and } |x - x_0| < \delta(\varepsilon, x_0) \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$ (3.1.1)

3.2. Finding $\delta(\varepsilon)$ for a given function at a given point

In this and the next section we will prove that some familiar functions are continuous.

A general strategy for proving that a given function $f$ is continuous at a given point $x_0$ is as follows:

Step 1. Simplify the expression $|f(x) - f(x_0)|$ and try to establish a simple connection with the expression $|x - x_0|$. The simplest connection is to discover positive constants $\delta_0$ and $K$ such that

$$x \in I \text{ and } x_0 - \delta_0 < x < x_0 + \delta_0 \Rightarrow |f(x) - f(x_0)| \leq K|x - x_0|.$$ (3.2.1)

Constants $\delta_0$ and $K$ might depend on $x_0$. Formulate your discovery as a lemma.

Step 2. Let $\varepsilon > 0$ be given. Use the result in Step 1 to define your $\delta(\varepsilon, x_0)$. For example, if (3.2.1) holds, then $\delta(\varepsilon, x_0) = \min\{\varepsilon/K, \delta_0\}$.

Step 3. Use the definition of $\delta(\varepsilon, x_0)$ from Step 2 and the lemma from Step 1 to prove the implication (3.1.1).

Example 3.2.1. We will show that the function $f(x) = x^2$ is continuous at $x_0 = 3$. Here $I = \mathbb{R}$ and we do not need to worry about the domain of $f$.

Step 1. First simplify

$$|f(x) - f(x_0)| = |x^2 - 3^2| = |(x + 3)(x - 3)| = |x + 3||x - 3|.$$ (3.2.2)
Now we notice that if \(2 < x < 4\) we have \(|x + 3| = x + 3 \leq 7\). Thus \((3.2.1)\) holds with \(\delta_0 = 1\) and \(K = 7\). We formulate this result as a lemma.

**Lemma.** Let \(f(x) = x^2\) and \(x_0 = 3\). Then
\[
|x - 3| < 1 \quad \Rightarrow \quad |x^2 - 3^2| < 7|x - 3|.
\]

**Proof.** Let \(|x - 3| < 1\). Then \(2 < x < 4\). Therefore \(x + 3 > 0\) and \(|x + 3| = x + 3 < 7\). By \((3.2.2)\) we now have \(|x^2 - 3^2| < 7|x - 3|\).

\[\square\]

Step 2. Now we define \(\delta(\epsilon) = \min\{\epsilon/7, 1\}\).

Step 3. It remains to prove \((3.1.1)\). To this end, assume \(|x - 3| < \min\{\epsilon/7, 1\}\). Then \(|x - 3| < 1\). Therefore, by Lemma we have \(|x^2 - 3^2| < 7|x - 3|\). Since by the assumption \(|x - 3| < \epsilon/7\), we have \(7|x - 3| < 7\epsilon/7\). Now the inequalities
\[
|x^2 - 3^2| < 7|x - 3| \quad \text{and} \quad 7|x - 3| < \epsilon
\]
imply that \(|x^2 - 3^2| < \epsilon\). This proves \((3.1.1)\) and completes the proof that the function \(f(x) = x^2\) is continuous at \(x_0 = 3\).

**Exercise 3.2.2.** Prove that the reciprocal function \(x \mapsto \frac{1}{x}, x \neq 0\), is continuous at \(x_0 = 1/2\).

**Exercise 3.2.3.** State carefully what it means for a function \(f\) not to be continuous at a point \(x_0\) in its domain. (Express this as a formal mathematical statement.)

**Exercise 3.2.4.** Consider the function \(f(x) = \text{sgn} x\). Find a point \(x_0\) at which the function \(f\) is not continuous. Provide a formal proof.

**Exercise 3.2.5.** Show that the function \(f(x) = x^2\) is continuous on \(\mathbb{R}\).

**Exercise 3.2.6.** Prove that \(q(x) = 3x^2 + 5\) is continuous on \(\mathbb{R}\).

### 3.3. Familiar continuous functions

**Exercise 3.3.1.** Let \(m, k \in \mathbb{R}\) and \(m \neq 0\). Prove that the linear function \(\ell(x) = mx + k\) is continuous on \(\mathbb{R}\).

**Exercise 3.3.2.** Let \(a, b, c \in \mathbb{R}\) and \(a \neq 0\). Prove that the quadratic function \(q(x) = ax^2 + bx + c\) is continuous on \(\mathbb{R}\).

**Exercise 3.3.3.** Let \(n \in \mathbb{N}\) and let \(x, x_0 \in \mathbb{R}\) be such that \(x_0 - 1 \leq x \leq x_0 + 1\). Prove the following inequality
\[
|x^n - x_0^n| \leq n(|x_0| + 1)^{n-1}|x - x_0|.
\]

**Hint:** First notice that the assumption \(x_0 - 1 \leq x \leq x_0 + 1\) implies that \(|x| < |x_0| + 1\). Then use the Mathematical Induction and the identity
\[
|x^{n+1} - x_0^{n+1}| = |x^{n+1} - x x_0^n + x_0^n - x_0^{n+1}|.
\]

**Exercise 3.3.4.** Let \(n \in \mathbb{N}\). Prove that the power function \(x \mapsto x^n, x \in \mathbb{R}\), is continuous on \(\mathbb{R}\).

**Exercise 3.3.5.** Let \(n \in \mathbb{N}\) and let \(a_0, a_1, \ldots, a_n \in \mathbb{R}\) with \(a_n \neq 0\). Prove that the \(n\)-th order polynomial
\[
p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n
\]
is a continuous function on \(\mathbb{R}\).
3.5. ALGEBRA OF CONTINUOUS FUNCTIONS

Exercise 3.3.6. Prove that the reciprocal function $x \mapsto \frac{1}{x}$, $x \neq 0$, is continuous on its domain.

Exercise 3.3.7. Prove that the square root function $x \mapsto \sqrt{x}$, $x \geq 0$, is continuous on its domain.

Exercise 3.3.8. Let $n \in \mathbb{N}$ and let $x$ and $a$ be positive real numbers. Prove that

$$\left| \sqrt[n]{x} - \sqrt[n]{a} \right| \leq \frac{\sqrt[n]{a}}{a} |x - a|.$$ 

HINT: Notice that the given inequality is equivalent to

$$b^{n-1} |y - b| \leq |y^n - b^n|, \quad y, b > 0.$$ 

This inequality can be proved using Exercise 2.7.7 (with $a = 1$ and $x = y/b$).

Exercise 3.3.9. Let $n \in \mathbb{N}$. Prove that the $n$-th root function $x \mapsto \sqrt[n]{x}$, $x \geq 0$, is continuous on its domain.

3.4. Various properties of continuous functions

Exercise 3.4.1. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$ and let $y$ be a real number such that $f(x_0) < y$. Then there exists $\alpha > 0$ such that

$$x \in I \cap (x_0 - \alpha, x_0 + \alpha) \quad \Rightarrow \quad f(x) < y.$$ 

Illustrate with a diagram.

Exercise 3.4.2. Let $f : I \to \mathbb{R}$ be a continuous function on $I$. Let $S$ be a non-empty bounded above subset of $I$ such that $u = \sup S$ belongs to $I$. Let $y \in \mathbb{R}$. Prove: If $f(x) \leq y$ for each $x \in S$, then $f(u) \leq y$.

3.5. Algebra of continuous functions

All exercises in this section have the same structure. With the exception of Exercise 3.5.3, there are three functions in each exercise: $f$, $g$ and $h$. The function $h$ is always related in a simple (green) way to the functions $f$ and $g$. Based on the given (green) information about $f$ and $g$ you are asked to prove a claim (red) about the function $h$.

Exercise 3.5.1. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x) + g(x), \quad x \in I.$$ 

(a) If $f$ and $g$ are continuous at $x_0 \in I$, then $h$ is continuous at $x_0$.

(b) If $f$ and $g$ are continuous on $I$, then $h$ is continuous on $I$.

Exercise 3.5.2. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x)g(x), \quad x \in I.$$ 

(a) If $f$ and $g$ are continuous at $x_0 \in I$, then $h$ is continuous at $x_0$.

(b) If $f$ and $g$ are continuous on $I$, then $h$ is continuous on $I$. 
Exercise 3.5.3. Let $g : I \to \mathbb{R}$ be a given function such that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{1}{g(x)}, \quad x \in I.$$ (a) If $g$ is continuous at $x_0 \in I$, then $h$ is continuous at $x_0$.
(b) If $g$ is continuous on $I$, then $h$ is continuous on $I$.

Exercise 3.5.4. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Assume that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x) g(x), \quad x \in I.$$ (a) If $f$ and $g$ are continuous at $x_0 \in I$, then $h$ is continuous at $x_0$.
(b) If $f$ and $g$ are continuous on $I$, then $h$ is continuous on $I$.

Exercise 3.5.5. Let $I$ and $J$ be non-empty subsets of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be given functions. Assume that the range of $f$ is contained in $J$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = g(f(x)), \quad x \in I.$$ (a) If $f$ is continuous at $x_0 \in I$ and $g$ is continuous at $f(x_0) \in J$, then $h$ is continuous at $x_0$.
(b) If $f$ is continuous on $I$ and $g$ is continuous on $J$, then $h$ is continuous on $I$.

3.6. Continuous functions on a closed bounded interval $[a, b]$

In this section we assume that $a, b \in \mathbb{R}$ and $a < b$.

Exercise 3.6.1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

HINT: Consider the set $W = \{ w \in [a, b) : \forall x \in [a, w] \ f(x) < 0 \}$.
Prove the following properties of $W$:
(i) $W$ does not have a maximum.
(ii) $W$ has a supremum. Set $w = \sup W$.
(iii) Review Exercise 3.4.2.
(iv) Connect the dots.

Exercise 3.6.2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$.

HINT: Consider the set $W = \{ v \in [a, b] : \exists z \in [a, b] \text{ such that } \forall x \in [a, v] \ f(x) < f(z) \}$.
Here $[a, a]$ denotes the set $\{a\}$. Prove the following properties of the set $W$:
(i) If $a < u$ and $[a, u] \subseteq W$ and there exists $t \in [a, b]$ such that $f(t) > f(u)$, then $u \in W$.
(ii) $W$ does not have a maximum.
(iii) $W$ has a supremum. Set $w = \sup W$ and prove $[a, w) \subseteq W$.
(iv) The items (ii) and (iii) yield information about $w$. 

Exercise 3.6.3. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then there exists \( d \in [a, b] \) such that \( f(d) \leq f(x) \) for all \( x \in [a, b] \).
Hint: Use Exercise 3.6.2.

Exercise 3.6.4. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then the range of \( f \) is a closed bounded interval.

Exercise 3.6.5. Consider the function \( f(x) = x^2, x \in \mathbb{R} \).
(a) Prove that 2 is in the range of \( f \).
(b) Prove that the range of \( f \) equals \([0, +\infty)\).

Definition 3.6.6. A function \( f \) is increasing on an interval \( I \) if \( x, y \in I \) and \( x < y \) imply \( f(x) < f(y) \). A function \( f \) is decreasing if \( x, y \in I \) and \( x < y \) imply \( f(x) > f(y) \). A function which is increasing or decreasing is said to be strictly monotonic.

Exercise 3.6.7. If \( f \) is continuous and increasing on \([a, b]\) or continuous and decreasing on \([a, b]\), then for each \( y \) between \( f(a) \) and \( f(b) \) there is exactly one \( x \in [a, b] \) such that \( f(x) = y \).

Exercise 3.6.8. Let \( f(x) = x^3 + x, x \in \mathbb{R} \). Prove that \( f \) has an inverse. That is, prove that for each \( y \in \mathbb{R} \) there exists unique \( x \in \mathbb{R} \) such that \( f(x) = y \).