ON THE MAXIMUM OF A CONTINUOUS FUNCTION

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In this note \(a\) and \(b\) are real numbers such that \(a < b\) and \(f : [a, b] \to \mathbb{R}\) is a function defined on \([a, b]\).

**Definition 1.** Let \(f : [a, b] \to \mathbb{R}\) be a given function. If \(z \in [a, b]\) and \(f(z) \geq f(x)\) for all \(x \in [a, b]\), then the value \(f(z)\) is called a maximum of \(f\).

**Definition 2.** A function \(f : [a, b] \to \mathbb{R}\) is continuous at a point \(x_0 \in [a, b]\) if for each \(\epsilon > 0\) there exists \(\delta = \delta(\epsilon, x_0) > 0\) such that
\[
\text{if } x \in [a, b] \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.
\]
The function \(f\) is continuous on \([a, b]\) if it is continuous at each point \(x_0 \in [a, b]\).

Let \(\alpha, \beta, \gamma \in [a, b], \alpha \leq \beta, \gamma\). In this note we say that the function \(f\) is dominated on \([\alpha, \beta]\) if there exists \(z_0 \in [a, b]\) such that \(f(x) < f(z_0)\) for all \(x \in [\alpha, \beta]\); see Fig. 1 and 2.

The following two lemmas give two simple properties of domination.

**Lemma 1.** Let \(\alpha, \alpha_1, \beta, \beta_1 \in [a, b]\) and \(\alpha \leq \alpha_1 \leq \beta_1 \leq \beta\). If \(f\) is dominated on \([\alpha, \beta]\), then \(f\) is dominated on \([\alpha_1, \beta_1]\).

**Proof.** If \(f\) is dominated on \([\alpha, \beta]\), then for some \(z_0 \in [a, b]\) we have \(f(x) < f(z_0)\) for all \(x \in [\alpha, \beta]\). Since \([\alpha_1, \beta_1] \subseteq [\alpha, \beta]\) we have \(f(x) < f(z_0)\) for all \(x \in [\alpha_1, \beta_1]\). Hence \(f\) is dominated on \([\alpha_1, \beta_1]\). \(\Box\)

**Lemma 2.** Let \(\alpha, \beta, \gamma \in [a, b]\) and \(\alpha \leq \beta \leq \gamma\). If \(f\) is dominated on both intervals \([\alpha, \beta]\) and \([\beta, \gamma]\), then \(f\) is dominated on the interval \([\alpha, \gamma]\).

**Proof.** Assume that \(f\) is dominated on both intervals \([\alpha, \beta]\) and \([\beta, \gamma]\). Then there exists \(z_0, z_1 \in [a, b]\) such that \(f(x) < f(z_0)\) for all \(x \in [\alpha, \beta]\) and \(f(x) < f(z_1)\) for all \(x \in [\beta, \gamma]\). Set
\[
z_2 := \begin{cases} z_0 & \text{if } f(z_1) \leq f(z_0), \\ z_1 & \text{if } f(z_0) < f(z_1). \end{cases}
\]
Then \(f(z_1) \leq f(z_2)\) and \(f(z_0) \leq f(z_2)\). Therefore, \(f(x) < f(z_2)\) for all \(x \in [\alpha, \gamma]\). Hence \(f\) is dominated on \([\alpha, \gamma]\). \(\Box\)

In the following three lemmas we prove properties of domination which require continuity of the function \(f\) at a point.

**Lemma 3.** Let \(d \in [a, b]\). If \(f\) is continuous at \(d\) and \(f(d)\) is not a maximum of \(f\), then there exists \(\eta > 0\) such that \(f\) is dominated on the interval \([d - \eta, d + \eta] \cap [a, b]\).

Lemma 5. Let $f$ be dominated on $[a, b]$. Assume

(i) $f$ is dominated on $[a, b]$ for every $\beta < d$;
(ii) $f$ is continuous at $d$;
(iii) $f(d)$ is not a maximum of $f$.

Then $f$ is dominated on $[a, d]$.

Proof. Assume (i), (ii) and (iii). By Lemma 3 there exists $\eta > 0$ such that $f$ is dominated on the interval $[d - \eta, d + \eta] \cap [a, b]$. Since $a < d$, the number $\nu = \min\{\eta, (d - a)/2\}$ is positive. By the definition of $\nu$ we have $a < d - \nu$ and thus $[d - \nu, d] \subseteq [d - \eta, d + \eta] \cap [a, b]$. Since $f$ is dominated on $[d - \eta, d + \eta] \cap [a, b]$, by Lemma 1 $f$ is also dominated on $[d - \nu, d]$. Since $a < d - \nu < d$, the assumption...
(i) implies that \( f \) is dominated on \([a, d - \nu]\). Since \( f \) is dominated on both intervals \([a, d - \nu]\) and \([d - \nu, d]\), Lemma 2 implies that \( f \) is dominated on \([a, d]\). \( \square \)

The next corollary is a partial contrapositive of the preceding lemma.

**Corollary 6.** Let \( d \in (a, b] \). Assume

(i) \( f \) is dominated on \([a, \beta]\) for every \( \beta < d \);
(ii) \( f \) is continuous at \( d \);
(iii) \( f \) is not dominated on \([a, d]\).

Then \( f(d) \) is a maximum of \( f \).

**Theorem.** Let \( a, b \in \mathbb{R} \), \( a < b \). Let \( f : [a, b] \to \mathbb{R} \) be a continuous function on \([a, b]\). Then there exists \( c \in [a, b] \) such that \( f(c) \geq f(x) \) for all \( x \in [a, b] \).

**Proof.** Case I. The value \( f(a) \) is a maximum of \( f \). In this case we can set \( c = a \).
Case II. The value $f(a)$ is not a maximum of $f$. Define (see Fig. 4)

$$W = \left\{ \beta \in [a, b] : f \text{ is dominated on } [a, \beta] \right\}.$$ 

Notice that $f$ is not dominated on $[a, b]$ since the statement

$$\exists z_0 \in [a, b] \text{ such that } \forall x \in [a, b] \quad f(x) < f(z_0)$$

is false. Therefore $b \notin W$.

Step 1. Since $f(a)$ is not a maximum of $f$, by Lemma 3 there exists $\eta_1 > 0$ such that $f$ is dominated on $[a, a + \eta_1] \cap [a, b]$. As $[a, b]$ is not dominated, $a + \eta_1 < b$. Thus $f$ is dominated on $[a, a + \eta_1] \cap [a, b] = [a, a + \eta_1]$. Hence $a + \eta_1 \in W$. Consequently, $W \neq \emptyset$. Since $W \subseteq [a, b]$, $W$ is bounded. Hence $c = \sup W$ exists by the Completeness Axiom. Since $b$ is an upper bound of $W$ and $a + \eta_1 \in W$, we have $a < c \leq b$.

Step 2. Let $\beta \in W$. Then $\beta \in [a, b)$ and $f$ is dominated on $[a, \beta]$. Since $f$ is continuous at $\beta$, Lemma 4 implies that there exists $\eta > 0$ such that $f$ is also dominated on $[a, \beta + \eta]$. Hence $\beta + \eta \in W$. This proves that $W$ does not have a maximum. Therefore $c \notin W$.

Step 3. Here we show that $[a, c) \subseteq W$. Let $\beta \in [a, c)$ be arbitrary. Since $\beta < c$ and $c = \sup W$, $\beta$ is not an upper bound of $W$. Hence, there exists $\gamma \in W$ such that $\beta < \gamma < c$. Since $f$ is dominated on $[a, \gamma]$ and $[a, \beta] \subseteq [a, \gamma]$, Lemma 1 implies that $f$ is dominated on $[a, \beta]$. Hence $\beta \in W$. This proves $[a, c) \subseteq W$.

Step 4. By Step 2, $c \notin W$. Therefore $f$ is not dominated on $[a, c]$. By Step 3 we have $[a, c) \subseteq W$. Therefore $f$ is dominated on $[a, \beta]$ for every $\beta \in [a, c)$. Now Corollary 6 implies that $f(c)$ is a maximum of $f$.

The proof is complete. $\square$