ON A ZERO OF A CONTINUOUS FUNCTION

BRANKO ĆURGUS

In this note $a$ and $b$ are real numbers and $a < b$.

Definition 1. A function $f : [a, b] \to \mathbb{R}$ is continuous at a point $x_0 \in [a, b]$ if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$x \in (x_0 - \delta, x_0 + \delta) \cap [a, b] \Rightarrow |f(x) - f(x_0)| < \epsilon.$$ 

Theorem. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

Proof. Assume $f(a) < 0$ and $f(b) > 0$.

Step 1. Set

$$W = \{x \in [a, b] : f(x) < 0\}.$$

Clearly $a \in W$, $b \not\in W$ and $W \subseteq [a, b)$. Therefore, $c = \sup W$ exists by the Completeness Axiom. Since $a \in W$ and $b$ is an upper bound for $W$ we have $c \in [a, b]$.
Step 2. Here we show that $W$ does not have a maximum. Let $u \in W$ be arbitrary. Then $u < b$ and $f(u) < 0$. Set $\epsilon_1 = -f(u)/2 > 0$. Since $\epsilon_1 > 0$ and $f$ is continuous at $u$ there exists $\delta_1 = \delta(\epsilon_1, u) > 0$ such that

\[ (1) \quad x \in [a, b] \cap (u - \delta_1, u + \delta_1) \quad \Rightarrow \quad f(u) - \epsilon_1 < f(x) < f(u) + \epsilon_1. \]

Set

\[ \mu = \frac{1}{2} \min \{\delta_1, b - u\}. \]

Then $\mu > 0$, $u + \mu < b$ and $u < u + \mu < u + \delta_1$. It follows from (1) that

\[ f(u + \mu) < f(u) + \epsilon_1 = f(u) + \left( -\frac{f(u)}{2} \right) = \frac{f(u)}{2} < 0. \]

Thus $u + \mu \in W$. Since $u + \mu > u$, we proved that $u$ is not a maximum of $W$.

Step 3. As $W$ does not have a maximum, $c \notin W$. Since $c \in [a, b]$ and $c \notin W$ we conclude that $f(c) \geq 0$.

Step 4. Here we show that $f(c) \leq 0$. Let $\epsilon > 0$ be arbitrary. Since $f$ is continuous at $c$, there exists $\delta = \delta(\epsilon, c) > 0$ such that

\[ (2) \quad x \in [a, b] \cap (c - \delta, c + \delta) \quad \Rightarrow \quad f(c) - \epsilon < f(x) < f(c) + \epsilon. \]

Since $c = \sup W$ and $\delta > 0$ there exists $w \in W$ such that

\[ c - \delta < w < c. \]

Now (2) and $f(w) < 0$ yield $f(c) - \epsilon < f(w) < 0$. Since $\epsilon > 0$ was arbitrary, we proved that $f(c) < \epsilon$ for all $\epsilon > 0$. Consequently $f(c) \leq 0$.

Step 5. In Step 3 we proved $f(c) \geq 0$. In Step 4 we proved $f(c) \leq 0$. Thus $f(c) = 0$. This completes the proof. \[ \square \]