1. **5 points** Find and simplify the derivatives of the following functions. \( f(x) = \frac{x e^x}{x^2 + 2} \)

\[
f'(x) = \frac{(e^x + xe^x)(x^2 + 2) - x e^x(2x)}{(x^2 + 2)^2} = \frac{e^x ((1 + x)(x^2 + 2) - 2x^2)}{(x^2 + 2)^2}
\]

\[
= \frac{e^x (x^2 + 2 + x^3 + 2x - 2x^2)}{(x^2 + 2)^2} = \frac{(x^3 - x^2 + 2x + 2) e^x}{(x^2 + 2)^2}
\]

2. **5 points** Find \( \int \left( \frac{3x^2}{x^3 + 1} \right) dx \).

\[
u = x^3 + 1 \quad du = 3x^2 \, dx
\]

\[
\int \left( \frac{3x^2}{x^3 + 1} \right) dx = \int \frac{du}{u} = \ln |u| + C = \ln |x^3 + 1| + C
\]

3. **5 points** Find \( \int \left( t^4 - 3 + \frac{2}{t^2} \right) dt \).

\[
\int \left( t^4 - 3 + \frac{2}{t^2} \right) dt = \int \left( t^4 - 3 + 2t^{-2} \right) dt = \frac{t^5}{5} - 3t + \frac{2t^{-1}}{-1} + C = \frac{t^5}{5} - 3t - \frac{2}{t} + C
\]

4. **5 points** A farmer has 600 feet of fencing with which she wants to enclose two adjacent fields. These two fields are of the same size and are to be separated by a fence using some of her 600 feet of fencing. What is the largest area that she can enclose with both fields?

\[
\text{Area} = A = xy
\]

Fencing = 600 = 2x + 3y \quad \text{So} \quad y = 200 - \frac{2}{3} x

\[
A = A(x) = x \left( 200 - \frac{2}{3} x \right) = 200x - \frac{2}{3} x^2
\]

\[
A'(x) = 200 - \frac{4}{3} x \quad \text{This is zero if} \quad x = 150
\]

For \( x = 150 \) \quad \( y = 200 - \frac{2}{3} \cdot 150 = 100 \)

The dimensions of maximum area are \( x = 150 \text{ ft} \) and \( y = 100 \text{ ft} \)

The largest area is 15,000 sq. ft.
5. 4 points Sketch the graph of a curve for which the following is true:
\[
f(-4) = 2 \quad f(-2) = -1 \quad f(1) = 3 \quad f(4) = 1 \quad f(7) = -3
\]
\[
f'(-2) = 0 \quad f'(4) = 0 \quad f'(1) \text{ is not defined}
\]
\[
f''(x) > 0 \text{ for } -4 < x < 1 \quad f''(x) > 0 \text{ for } 1 < x < 4 \quad f''(x) < 0 \text{ for } 4 < x < 7
\]

6. 5 points Find the value of \(x\) and the value of \(y\) at the absolute minimum and absolute maximum of the function
\[
f(x) = 4x^3 - 15x^2 + 12x + 5 \text{ on the interval } [1, 4].
\]
\[
f'(x) = 12x^2 - 30x + 12 = 6(2x^2 - 5x + 2) = 6(2x - 1)(x - 2)
\]

<table>
<thead>
<tr>
<th>reason</th>
<th>(x)</th>
<th>(f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>end point</td>
<td>1</td>
<td>(4 - 15 + 12 + 5 = 6)</td>
</tr>
<tr>
<td>end point</td>
<td>4</td>
<td>(44^3 - 154^2 + 124 + 5 = 69)</td>
</tr>
<tr>
<td>extreme pt.</td>
<td>2</td>
<td>(42^3 - 152^2 + 122 + 5 = 1)</td>
</tr>
<tr>
<td>extreme pt.</td>
<td>(1/2)</td>
<td>not in domain of (f)</td>
</tr>
</tbody>
</table>

The absolute maximum is at \((4, 69)\) The absolute minimum is at \((2, 1)\)

7. 5 points Find the absolute minimum of the function \(f(x) = 5x - 2x \ln x\) over the interval \(2 \leq x \leq 8\).

\[
f'(x) = 5 - 2 \ln x - 2x \frac{1}{x} = 5 - 2 \ln x - 2 = 3 - 2 \ln x
\]

This is zero is \(x = e^{3/2} = 4.48\). If \(x\) is bigger than this number, \(f'\) is negative; whereas if \(x\) is less than this number \(f'\) is positive. So we have a relative maximum when \(x = e^{3/2}\)

<table>
<thead>
<tr>
<th>reason</th>
<th>(x)</th>
<th>(f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>end point</td>
<td>2</td>
<td>(10 - 4 \ln 2 = 7.23)</td>
</tr>
<tr>
<td>end point</td>
<td>8</td>
<td>(40 - 16 \ln 8 = 6.73)</td>
</tr>
<tr>
<td>extreme pt.</td>
<td>(e^{3/2})</td>
<td>(5e^{3/2} - 2e^{3/2} \ln e^{3/2} = 8.96)</td>
</tr>
</tbody>
</table>

The absolute minimum is at \((8, 6.73)\)
8. 5 points Find the values of \( x \) for which \( f(x) = \frac{x^2}{e^x} \) is increasing.

\[
f'(x) = \frac{2x e^x - x^2 e^x}{(e^x)^2} = \frac{x(2 - x)}{e^x}
\]

This is positive, so \( f \) is increasing, when \( x \) is between 0 and 2.

9. 5 points Find \( \int 6e^{0.5x} \, dx \)

Note typo on the exam. An \( x \) was missing

\[
u = 0.5 \, x \quad du = 0.5 \, dx
\]

\[
\int 6e^{0.5x} \, dx = 6 \int e^{0.5x} 2 \cdot 0.5 \, dx = 12 \int e^u \, du = 12e^u + C = 12e^{0.5x} + C
\]

10. 5 points The price-demand equation for \( x \) units of a commodity is \( p(x) = 500e^{-0.025x} \). Find the production level and price per unit that produces the maximum revenue. What is that maximum revenue?

Revenue \( R(x) = px = 500xe^{-0.025x} \)

So \( R'(x) = 500e^{-0.025x} + 500xe^{-0.025x}(-0.025) = 500(1 - 0.025x)e^{-0.025x} \)

This is zero when \( x = \frac{1}{0.025} = 40 \) To maximize revenue produce 40

The corresponding price will be \( p(40) = 500e^{-0.025(40)} = 183.94 \) and

the maximum revenue will be \( 40(183.94) = 7357.59 \)

11. 5 points For what values of \( x \) is the graph of \( f(x) = \ln(1 + x^2) \) concave up?

\[
f'(x) = \frac{1}{1 + x^2} \cdot 2x = \frac{2x}{1 + x^2}.
\]

\[
f''(x) = \frac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2} = \frac{2(1 - x)(1 + x)}{(1 + x^2)^2}
\]

This is positive for \(-1 < x < 1\) so \( f \) is concave up for \( x \) between \(-1\) and 1.